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Jian HUANG
Jiangxi University of Finance and Economics

Mingming LENG
Department of Computing and Decision Sciences, Lingnan University

Zhengde DAI
Yunnan University

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Novel Homoclinic and Heteroclinic Solutions for the 2D Complex Cubic Ginzburg-Landau Equation[1]

Jian Huang[2, 3], **Mingming Leng**[4] and **Zhengde Dai**[5]

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[2]Corresponding author (Email: jianhuangvictor@yahoo.com.cn; Telephone: +86-791-389-1471).

[3]School of Information Management, Jiangxi University of Finance and Economics, Nanchang, Jiangxi 330013, China.

[4]Department of Computing and Decision Sciences, Lingnan University, 8 Castle Peak Road, Tuen Mun, Hong Kong.

[5]School of Mathematics and Physics, Yunnan University, Kunming, Yunnan 650091, China.

Abstract

Homoclinic and heteroclinic solutions are two important concepts that are used to investigate the complex properties of nonlinear evolutionary equations. In this paper, we perform hyperbolic and linear stability analysis, and prove the existence of homoclinic and heteroclinic solutions for two-dimensional cubic Ginzburg-Landau equation with periodic boundary condition and even constraint. Then, using the Hirota's bilinear transformation, we find the closed-form homoclinic and heteroclinic solutions. Moreover, we find that the homoclinic tubes (which are formed by a pair of symmetric homoclinic solutions) and two families of heteroclinic solutions are asymptotic to a periodic cycle in one dimension.

Keywords: 2D cubic Ginzburg-Landau equation; homoclinic orbits; heteroclinic orbits; hyperbolic property; linearized stability; Hirota's bilinear transformation.

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1 Introduction

The existence of homoclinic and heteroclinic solutions are important for investigating the complex dynamics of partial differential equations. In recent years, a number of methods have been developed to prove the existence of homoclinic solutions in some nonlinear evolutionary equations (NEE) such as nonlinear Schrödinger Equation [1], Sine-Gordon equation[2], long-short wave equation [3], DS II equation [4], Boussinesq equations [5], etc. Furthermore, researchers recently found a novel method which can be used to analyze the homoclinic solutions for Davey-Stewartzon equations [4, 5], Boussinesq equation [6], Sine-Gordon equation [7], Zakharov equation [8], etc. To use the novel method for proving the existence of homoclinic/heteroclinic solutions of NEE, we commonly adopt the following two steps: In the first step, we need to prove that the fixed points or cycles of NEE are hyperbolic, which shows that the fixed points or cycles are saddle points or cycles. In the second step, we should perform linearized stability analysis, in order to demonstrate that the fixed points or cycles are linearly unstable. Then, we can use the Hirota's bilinear transformation to derive the explicit homoclinic/heteroclinic solutions.

In the nonlinear science field, the exact homoclinic and heteroclinic solutions are important to the analysis of complex dynamics for NEE. Among various NEE, the complex Ginzburg-Landau equation (CGLE) is an important and well-known system. In this paper, we mainly focus on the homoclinic and heteroclinic solutions of 2D cubic CGLE with periodic boundary condition and even constraint.

The CGLE is an important system in the area of nonlinear optics that describes the propagation of optical pulses in optic fibers. Since the CGLE system appeared, the solutions of CGLE have been extensively examined from many different perspectives; see, e.g., [9–12]. Sakaguchi and Boris [13] proposed a new model that describes the nonlinear planer waveguide incorporated into a closed optical cavity, and presented a CGLE with an anisotropy of a novel type which is diffractive in one direction and diffusive in the other. Dai *et al.* [14] has examined the 2D Ginzburg-Landau equation which is similar to that was developed by Sakaguchi and Boris [13]. However, our paper differs from [14] because of the following facts: (i) We consider Sakaguchi and Malomed's CGLE system [13] which is different from that in [14]. (ii) We prove the existence of the homoclinic and heteroclinic solutions, and also derive the closed-form results of these two solutions. But, Dai *et al.* [14] only found the closed-form homoclinic solution. (iii) We investigate the structures of the homoclinic and heteroclinic solutions, which were not considered by Dai *et al.* [14].

The CGLE system developed by Sakaguchi and Malomed [13] can be written as follows:

$$u_t = k_1 u + (k_2 + ik_3)u_{xx} + (k_2 + ik_3)u_{yy} - (k_4 + ik_5)|u|^2 u, \tag{1}$$

with boundary condition of period $(2\pi/p_1, 2\pi/p_2)$ and even constraint; that is,

$$u(t, x + 2\pi/p_1, y + 2\pi/p_2) = u(t, x, y) \text{ and } u(t, -x, -y) = u(t, x, y),$$

where k_1, k_2, k_3, k_4 and k_5 are real constants; p_1 and p_2 are real constants that should be determined. Obviously, $e^{i(at+\varphi)}$ is a solution of (1) and also is a fixed point, where φ is a real constant, $k_1 = k_4$ and $a = -k_5$. There thus exists a fixed circle consisting of infinite number of fixed points with different φ. To the best of our knowledge, very few closed-form solutions have been found for this CGLE so far.

The remainder of this paper is organized as follows: In Section 2, we use the novel method that was developed in [4–8], we prove the existence of heteroclinic and homoclinic solutions. In Section 3 we derive the explicit heteroclinic and homoclinic solutions. The paper ends with a further discuss of heteroclinic and homoclinic solutions.

2 Hyperbolic Property and Linearized Stability Analysis

Before exactly deriving the analytical homoclinic/heteroclinic solutions, we should prove the existence of these two types of solutions. we commonly adopt the hyperbolic property analysis to show that the fixed point/circle is saddle point/circle and adopt the linearized stability analysis to determine the number of the unstable modes. We summarize the main results as the following two lemmas.

Proposition 1 If $3k_5^2 + k_2^2(k^2 + k'^2)^2 + k_3^2(k^2 + k'^2)^2 + 2k_1k_2(k^2 + k'^2) + 4k_3k_5(k^2 + k'^2) < 0$, then the fixed circle e^{iat} is a saddle circle.

Proof. Assume that $u(x, y, t) = u_1(x, y, t) + iu_2(x, y, t)$. Under the assumption, we separate the real and image parts, and arrive to the following equations

$$\begin{cases} u_{1t} = k_1u_1 + k_2u_{1xx} - k_3u_{2xx} + k_2u_{1yy} - k_3u_{2yy} - k_4(u_1^2 + u_2^2)u_1 + k_5(u_1^2 + u_2^2)u_2, \\ u_{2t} = k_1u_2 + k_3u_{1xx} + k_2u_{2xx} + k_3u_{1yy} + k_2u_{2yy} - k_5(u_1^2 + u_2^2)u_1 - k_4(u_1^1 + u_2^2)u_2. \end{cases}$$

We linearize the above equations, and find that

$$\begin{cases} U_{1t} = & k_1U_1 + k_2U_{1xx} - k_3U_{2xx} + k_2U_{1yy} - k_3U_{2yy} - k_4(2u_1U_1 + 2u_2U_2)u_1 - \\ & k_4(u_1^2 + u_2^2)U_1 + k_5(2u_1U_1 + 2u_2U_2)u_2 + k_5(u_1^2 + u_2^2)U_2, \\ U_{2t} = & k_1U_2 + k_3U_{1xx} + k_2U_{2xx} + k_3U_{1yy} + k_2U_{2yy} - k_5(2u_1U_1 + 2u_2U_2)u_1 - \\ & k_5(u_1^2 + u_2^2)U_1 - k_4(2u_1U_1 + 2u_2U_2)u_2 - k_4(u_1^2 + u_2^2)U_2, \end{cases} \quad (2)$$

where $k_1 = k_4$. Since $u(x, t) = e^{iat}$ is a fixed point circle, we find that $u_1 = \cos at$ and $u_2 = \sin at$. We consider the following simple case: only one wave number k (k') in the x direction (y direction) and U_j ($j = 1, 2$) is the eigenfunction of spatial operators (∂_{xx}, ∂_{yy}) around the fixed point circle, which has the form

$$U_{jxx} = -k^2U_j \text{ and } U_{jyy} = -k'^2U_j. \quad (3)$$

Using (3), we solve (2) for the eigenvalues of the coefficient matrix, and find that, if

$$3k_5^2 + k_2^2(k^2 + k'^2)^2 + k_3^2(k^2 + k'^2)^2 + 2k_1k_2(k^2 + k'^2) + 4k_3k_5(k^2 + k'^2) < 0, \tag{4}$$

then the eigenvalues are

$$\lambda = -k_4 - k_2(k^2 + k'^2) \pm \sqrt{k_4^2 - 3k_5^2 - k_3^2(k^2 + k'^2)^2 - 4k_3k_5(k^2 + k'^2)}.$$

Therefore, we find that an eigenvalue is positive but the other is negative; this proves the existence of saddle points. That is, fixed point solution e^{iat} is a saddle point circle. ■

Remark 1 The condition (4) is satisfied when k_2 and k_4 are sufficiently small and $|k_5|$ is sufficiently large when $k_5 < 0$. For example, when $k^2 + k'^2 = 1$ and $k_5 = k_3/2$, the condition (4) is satisfied. $\triangleleft$

Proposition 2 If $a = -k_5$, then we can obtain

$$0 < N < \frac{1}{p_2}\sqrt{\frac{\sqrt{k_4^2 + k_5^2} - k_5}{k_3(\gamma_1^2 + 1)}}, \tag{5}$$

where N is independent of t and represents the number of the unstable modes. In fact, the number N determines the complexity of the homoclinic structure. Hence, the fixed point is hyperbolic.

Proof. Consider a small perturbation of the following form:

$$u(x, y, t) = \bar{u}(x, y, t)[1 + \varepsilon(x, y, t)], \tag{6}$$

where $\bar{u} = e^{i(at+\varphi)}$ and $|\varepsilon| \ll 1$. Substituting (6) into (1) and keeping linear terms of ε yields the following linearized equation

$$\varepsilon_t = (k_2 + ik_3)\varepsilon_{xx} + (k_2 + ik_3)\varepsilon_{yy} - (k_4 + ik_5)\varepsilon^* + [k_1 - 2(k_4 + ik_5) - ia]\varepsilon, \tag{7}$$

where the superscript "*" denotes the complex conjugate. In order to find the solution of the form

$$\varepsilon = Ae^{i\mu_n x + i\bar{\mu}_n y + \sigma_n t} + Be^{-i\mu_n x - i\bar{\mu}_n y + \sigma_n t},$$

where A and B are complex constants; $\mu_n = p_1 n$ and $\bar{\mu}_n = p_2 n$; σ_n is the growth rate of nth mode, we need the following equations

$$\begin{cases} (\sigma_n + k_2\mu_n^2 + ik_3\mu_n^2 + k_2\bar{\mu}_n^2 + ik_3\bar{\mu}_n^2 + 2k_4 + 2ik_5 + ia - k_1)A + (k_4 + ik_5)B^* = 0 \\ (\sigma_n + k_2\mu_n^2 + ik_3\mu_n^2 + k_2\bar{\mu}_n^2 + ik_3\bar{\mu}_n^2 + 2k_4 + 2ik_5 + ia - k_1)B + (k_4 + ik_5)A^* = 0 \end{cases} \tag{8}$$

Solving equation (8) for σ_n, we have

$$\sigma_n = -k_2\mu_n^2 - k_2\bar{\mu}_n^2 - 2k_4 + k_1 \pm \sqrt{k_4^2 + k_5^2 - (k_3\mu_n^2 + k_3\bar{\mu}_n^2 + 2k_5 + a)^2}.$$

Assuming that $p_1 = \gamma_1 p_2$ and $\gamma_1 \neq 1$, we find that

$$0 < N < \frac{1}{p_2}\sqrt{\frac{\sqrt{k_4^2 + k_5^2} - 2k_5 - a}{k_3(\gamma_1^2 + 1)}},$$

because $\mu_n = p_1 n$ and $\bar{\mu}_n = p_2 n$. When $a = -k_5$, we can obtain the inequality (5). ■

The hyperbolic property (in Proposition 1) and our linearized stability analysis (in Proposition 2) ensure the existence of homoclinic/heteroclinic solutions for CGLE.

3 Closed-Form Homoclinic and Heteroclinic Solutions

In this section we use our analytical results in Section 2 to derive the closed-form homoclinic and heteroclinic solutions.

Theorem 1 The CGLE (1) has two families of the closed-form homoclinic and heteroclinic solutions such as

$$u_1 = e^{iat}\frac{1 + 2b_1\cos(p_1x + p_2y)e^{\Omega_1 t+\gamma} + b_3 e^{2\Omega_1 t + 2\gamma}}{1 + 2b_4\cos(p_1x + p_2y)e^{\Omega_1 t+\gamma} + b_5 e^{2\Omega_1 t + 2\gamma}}, \tag{9}$$

and

$$u_2 = e^{iat}\frac{1 + 2b_1\cos(p_1x + p_2y)e^{\Omega_2 t+\gamma} + b_3 e^{2\Omega_2 t + 2\gamma}}{1 + 2b_4\cos(p_1x + p_2y)e^{\Omega_2 t+\gamma} + b_5 e^{2\Omega_2 t + 2\gamma}}, \tag{10}$$

when the parameters satisfy the following relations

$$\begin{cases} \lambda = -k_1 - ik_5, \\ b_1 = b_2 = \dfrac{\Omega - (k_2 + ik_3)(p_1^2 + p_2^2)}{\Omega + (k_2 + ik_3)(p_1^2 + p_2^2)} b_4, \\ b_3 = \left[\dfrac{\Omega - (k_2 + ik_3)(p_1^2 + p_2^2)}{\Omega + (k_2 + ik_3)(p_1^2 + p_2^2)}\right]^2 b_5, \\ b_5 = \dfrac{\Omega^2 - (k_2 + ik_3)^2(p_1^2 + p_2^2)^2}{\Omega^2} b_4^2, \\ p_1 = \gamma_1 \sin\phi,\ p_2 = \sin\phi, \end{cases} \tag{11}$$

and

$$\Omega_{1,2} = -(k_1 + k_2(p_1^2 + p_2^2)) \pm \sqrt{k_3^2 - 2k_3k_5(p_1^2 + p_2^2) - k_3^2(p_1^2 + p_2^2)^2}, \tag{12}$$

where

$$p_2^2 < \frac{1}{\gamma_1^2 + 1}\left[\frac{\sqrt{k_3^2 + k_5^2} - k_5}{k_3}\right]. \tag{13}$$

Proof. We can easily find that the system (1) has the following plane wave solution:

$$u = e^{-ik_5t}v(x, y, t). \tag{14}$$

Thus, using (14), we can transform (1) into the following system

$$v_t = (k_2 + ik_3)v_{xx} + (k_2 + ik_3)v_{yy} + (k_1 + ik_5)v - (k_4 + ik_5)|v|^2v. \tag{15}$$

By the dependent variable transformation $v = G/F$, the system (15) can be transformed into the following bilinear form

$$\begin{cases} [D_t - (k_2 + ik_1)(D_x^2 + D_y^2)]G \cdot F - (k_1 + ik_5 + \lambda)GF = 0, \\ (k_2 + ik_3)F^*(D_x^2 + D_y^2)F \cdot F + (k_1 + ik_5)FGG^* + \lambda FFF^* = 0. \end{cases} \tag{16}$$

where λ is a complex constant which shall be determined later, and the Hirota's bilinear operator $D_x^m D_t^k$ [15, 16] is defined as,

$$D_x^m D_t^k a \cdot b = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^k a(x, t)b(x', t')|_{x'=x,t'=t}.$$

Assume G and F have the following forms

$$\begin{cases} G = 1 + (b_1e^{ip_1x+ip_2y} + b_2e^{-ip_1x-ip_2y})e^{\Omega t+\gamma} + b_3e^{2\Omega t+2\gamma}, \\ F = 1 + b_4(e^{ip_1x+ip_2y} + e^{-ip_1x-ip_2y})e^{\Omega t+\gamma} + b_5e^{2\Omega t+2\gamma}, \end{cases} \tag{17}$$

where a, p, Ω and γ are real, and b_1, b_2, b_3, b_4 and b_5 are complex. Substituting (17) into (16), we find the parameter relations (11) and

$$\Omega^2 + 2(k_1 + k_2(p_1^2 + p_2^2))\Omega + 2(p_1^2 + p_2^2)(k_1k_2 + k_3k_5) + (k_2^2 + k_3^2)(p_1^2 + p_2^2)^2 = 0 \tag{18}$$

From equation (18), we find two solutions for parameter Ω such as (12) and the condition (13) assure that $\Omega_{1,2}$ are real. ∎

When $k_2 = 0$, the solutions given by (9) and (10) represent orbits homoclinic to the fixed periodic circles. More precisely, we find that

$$u_1 \to e^{iat} \quad \text{and} \quad u_2 \to e^{iat}\left[\frac{\Omega - ik_3(p_1^2 + p_2^2)}{\Omega + ik_3(p_1^2 + p_2^2)}\right]^2, \quad \text{as } t \to -\infty,$$

and

$$u_1 \to e^{iat}\left[\frac{\Omega - ik_3(p_1^2 + p_2^2)}{\Omega + ik_3(p_1^2 + p_2^2)}\right]^2 \quad \text{and} \quad u_2 \to e^{iat}, \quad \text{as } t \to +\infty.$$

Hence, there exists a phase shift between the homoclinic solutions u_1 and u_2. That is, if $u_1(x_0, y_0, t)$ is a homoclinic solution, then $u_1(x_0 + \pi/p_1, y_0 + \pi t/p_2)$ is another homoclinic solution. As a result,

u_1 and u_2 form a symmetric pair of homoclinic solutions which shapes homoclinic tubes.

When $k_2 \neq 0$, the solutions given by (9) and (10) represent orbits heteroclinic to the fixed periodic circles; that is,

$$u_1 \to e^{iat} \quad \text{and} \quad u_2 \to e^{iat}\left[\frac{\Omega - (k_2 + ik_3)(p_1^2 + p_2^2)}{\Omega + (k_2 + ik_3)(p_1^2 + p_2^2)}\right]^2, \quad \text{as } t \to -\infty,$$

and

$$u_1 \to e^{iat}\left[\frac{\Omega - (k_2 + ik_3)(p_1^2 + p_2^2)}{\Omega + (k_2 + ik_3)(p_1^2 + p_2^2)}\right]^2 \quad \text{and} \quad u_2 \to e^{iat}, \quad \text{as } t \to +\infty.$$

Therefore, there also exist a phase shift between the heteroclinic solutions u_1 and u_2. If $u_1(x_0, y_0, t)$ is a heteroclinic solution, then $u_1(x_0 + 2\pi/p_1, y_0 + 2\pi/p_2, t)$ is another heteroclinic solution. Thus, u_1 and u_2 are a symmetric pair of heteroclinic solutions and all of these orbits form the heteroclinic tubes.

4 Further Discussion

In what follows, we discuss the structure of the homoclinic solutions u_1. (Note that the discussion for u_2 is similar to that for u_1.) Assume the following transformation

$$\frac{\Omega_1 - ik_3(p_1^2 + p_2^2)}{\Omega_1 + ik_3(p_1^2 + p_2^2)} = \frac{\Omega_1^2 - 2ik_3\Omega_1(p_1^2 + p_2^2) - k_3^2(p_1^2 + p_2^2)^2}{\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2} = X + Yi.$$

Then, we separate the real and imaginary part as follows,

$$X = \frac{\Omega_1^2 - k_3^2(p_1^2 + p_2^2)^2}{\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2} \quad \text{and} \quad Y = \frac{-2k_3\Omega_1(p_1^2 + p_2^2)}{\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2}.$$

We thus arrive to the following equation,

$$u_1 = e^{iat}\frac{1 + 2(X + Yi)b_4 \cos{(p_1x + p_2y)}e^{\Omega_1 t + \gamma} + (X + Yi)^2 b_5 e^{2\Omega_1 t + 2\gamma}}{1 + 2b_4 \cos(p_1x + p_2y)e^{\Omega_1 t + \gamma} + b_5 e^{2\Omega_1 t + 2\gamma}}.$$

Letting $Z^2 \equiv (\Omega_1^2 + k_3^2 p^4)/\Omega_1^2$, we have,

$$u_1 = e^{iat}\frac{1 + 2(X + Yi)b_4 \cos(p_1x + p_2y)e^{\Omega_1 t + \gamma} + [(X + Yi)Zb_4 e^{\Omega_1 t + \gamma}]^2}{1 + 2b_4 \cos(p_1x + p_2y)e^{\Omega_1 t + \gamma} + [Zb_4 e^{\Omega_1 t + \gamma}]^2}.$$

It is easy to find that u_1 is a smooth plane with $(1+2)-$dimension which depends on variables

$$(x, y, t) \in \left(\frac{2n\pi}{p_1}, \frac{2(n+1)\pi}{p_1}\right) \times \left(\frac{2n\pi}{p_2}, \frac{2(n+1)\pi}{p_2}\right) \times (-\infty, +\infty),$$

for $n = \cdots, -2, -1, 0, 1, 2, \cdots$, and the parameters space

$$\begin{aligned}(a, p_1, p_2) \in \quad & R^+ \times \left[-\sqrt{\frac{1}{\gamma_1^2+1}\left[\frac{\sqrt{k_3^2+k_5^2}-k_5}{k_3}\right]}, \sqrt{\frac{1}{\gamma_1^2+1}\left[\frac{\sqrt{k_3^2+k_5^2}-k_5}{k_3}\right]} \right] \\ & \times \left[-\sqrt{\frac{1}{\gamma_1^2+1}\left[\frac{\sqrt{k_3^2+k_5^2}-k_5}{k_3}\right]}, \sqrt{\frac{1}{\gamma_1^2+1}\left[\frac{\sqrt{k_3^2+k_5^2}-k_5}{k_3}\right]} \right].\end{aligned}$$

Assuming $u_1 = \rho_1(x, y, t)e^{i\theta(x,y,t)}$, we can show that

$$\begin{aligned}\rho_1^2 \quad = \quad & \frac{a^2}{G}[1 + 4X\cos(p_1x + p_2y)E(t) + 2(X^2 - Y^2)Z^2E^2(t) + 4\cos^2(p_1x + p_2y)E^2(t) \\ & 4X(X^2 + Y^2)Z^2E^3(t)\cos(p_1x + p_2y) + (X^2 + Y^2)^2Z^4E^4(t)],\end{aligned}$$

where

$$\begin{aligned}G(t) \quad \equiv \quad & 1 + 4\cos(p_1x + p_2y)E(t) + 4\cos^2(p_1x + p_2y)E^2(t) \\ & +2Z^2E^2(t) + 4Z^2\cos(p_1x + p_2y)E^3(t) + Z^4E^4(t),\end{aligned}$$

and

$$E(t) = b_4e^{\Omega_1 t+\gamma}.$$

Since $X^2 + Y^2 = 1$, we find that

$$\begin{aligned}\rho_1^2 \quad = \quad & \frac{a^2}{G}[1 + 4X\cos(p_1x + p_2y)E(t) + 2(X^2 - Y^2)Z^2E^2(t) + 4\cos^2(p_1x + p_2y)E^2(t) \\ & +a^2[4XZ^2E^3(t)\cos(p_1x + p_2y) + Z^4E^4(t)],\end{aligned} \tag{19}$$

Note that

$$X^2 - Y^2 = \frac{[(p_1^2 + p_2^2 - 2\Omega_1)^2 - 8\Omega_1^2][(p_1^2 + p_2^2 + 2\Omega_1)^2 - 8\Omega_1^2]}{(p_1^2 + p_2^2)^2 + 4\Omega_1^2}.$$

As a result, when $|t| \longrightarrow \infty$, we find that

$$\rho_1^2 = \frac{a^2[Z^2E(t) + 4X\cos(p_1x + p_2y)] + \varepsilon_1(x, y, t)}{Z^2E(t) + 4\cos(p_1x + p_2y)] + \varepsilon_2(x, y, t)}, \quad \text{for } t > 0;$$

and

$$\rho_1^2 = \frac{a^2[1 + 4XE(t)\cos(p_1x + p_2y)] + \varepsilon_3(x, y, t)}{1 + 4E(t)\cos(p_1x + p_2y) + \varepsilon_4(x, y, t)}, \quad \text{for } t < 0,$$

where, as $t \to \infty$, $|\varepsilon_i(x, y, t)| \ll 1$ for arbitrary x, y.

Assuming $\rho_1^2 = a^2$, we can use (19) to attain the following equation,

$$2Z^2(X - 1)\cos(p_1x + p_2y)E^2(t) + Z^2(X^2 - Y^2 - 1)E(t) + 2(X - 1)\cos(p_1x + p_2y) = 0,$$

which can be re-written, by using the expressions of X, Y and Z^2, as,

$$(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)\cos(p_1x + p_2y)E^2(t) + \Omega_1^2 E(t) + 4\Omega_1^2\cos(p_1x + p_2y) = 0. \tag{20}$$

From (20), we obtain

$$E(t) = \frac{-\Omega_1^2 \pm \Omega_1^2\sqrt{1 - \dfrac{16}{\Omega_1^2}\cos^2(p_1x + p_2y)(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)}}{2(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)\cos(p_1x + p_2y)}. \tag{21}$$

Note $E(t) = b_4e^{\Omega_1 t + \gamma}$. when $b_4 > 0$, we have $E(t) > 0$. According to the above expression (21), in order to assure that $E(t) > 0$, if $\cos(p_1x + p_2y) < 0$, we then have the following expressions for $E(t)$:

$$E(t) = \frac{-\Omega_1^2 + \Omega_1^2\sqrt{1 - \dfrac{16}{\Omega_1^2}\cos^2(p_1x + p_2y)(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)}}{2(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)\cos(p_1x + p_2y)}, \quad \text{if } t > 0 \tag{22}$$

and

$$E(t) = \frac{-\Omega_1^2 - \Omega_1^2\sqrt{1 - \dfrac{16}{\Omega_1^2}\cos^2(p_1x + p_2y)(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)}}{2(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)\cos(p_1x + p_2y)}, \quad \text{if } t < 0 \tag{23}$$

When $t > 0$, if $\cos(p_1x + p_2y) < 0$ and $|\cos(p_1x + p_2y)| \ll 1$, we then find that $E(t)$ in (22) approaches infinity, i.e., $E(t) \to +\infty$. When $t < 0$ and $t \to -\infty$, we choose x, y to get the follows equation

$$\sqrt{\frac{16}{\Omega_1^2}\cos^2(p_1x + p_2y)(\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2)} = \varepsilon.$$

Then, we compute $E(t)$ in (23) as,

$$E(t) = \frac{4\Omega_1(1 - \sqrt{1 - \varepsilon^2})}{\sqrt{\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2}\varepsilon} = \frac{4\Omega_1\varepsilon}{\sqrt{\Omega_1^2 + k_3^2(p_1^2 + p_2^2)^2}(1 + \sqrt{1 - \varepsilon^2})}.$$

As a result, when $\varepsilon \to 0$, $E(t) \to 0$. On one hand, we learn from the above argument that the homoclinic flows are periodic about (x, y) and across the plane $p + 1$ and vibrate in the small region when (x, y) varies. One the other hand, as $t \to \infty$,

$$\theta(x, y, t) = at + \arctan\frac{2YE(t)\cos(p_1x + p_2y) + 2XYZ^2E^2(t)}{1 + 2XE(t)\cos(p_1x + p_2y) + (X^2 - Y^2)Z^2E^2(t)} \to \infty,$$

which shows that, when we fix (x, y), the orbits circulates the phase space for infinite times. Note that we don't perform the analysis for u_2, because it is similar to our above analysis for u_1.

Since our analysis for the heteroclinic flows is similar to the above for the homoclinic flows, we don't analyze the structure of the heteroclinic flows here.

5 Concluding Remarks

In this paper, we first proved the existence of homoclinic and heteroclinic solutions for the 2D cubic Ginzburg-Landau equation. Then, by using the Hirota's bilinear method, we computed the closed-form homoclinic and heteroclinic solutions. Moreover, we discussed the structure of the homoclinic and heteroclinic solutions. In future, we may consider the following interesting problems: What is the relation between the homoclinic (or heteroclinic) solutions and the chaos phenomenon arising in CGLE? Whether do there exist the homoclinic and heteroclinic solutions for the $n-$dimensional CGLE systems ($n \geq 3$)?

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