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Product Boundary, vertical Competition, and the double Mark-up problem

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We develop a model in which a main product (called product A) provides a performance quality z by itself, whereas a complementary product (called product B) is useless by itself but enhances the main product's performance quality to q >z. This asymmetric complementarity gives rise to the following results. First, if z is relatively small, then firms A and B behave as if the products are symmetrically complementary with the usual double marginalization problem. Second, if z is sufficiently large, then firms A and B price their products as if they are independent. Third, over a certain range of intermediate z, no pure-strategy Nash equilibrium exists.

1. Introduction

In the computing industry, since 1990 there has been no single dominant vertically integrated firm. Instead, the industry is characterized by vertical disintegration i.e., computer systems or platforms consist of many vertically related layers of components. Firms in different layers rely on one another, but at the same time they compete against each other for a bigger share of the industry profits. It is important to understand complementarity among different components.

In 1838, Cournot analysed the pricing of symmetrically complementary products, like left and right shoes, and identified the well known "double mark-up problem," i.e., when the two complementary products are supplied by two independent monopolies, the prices are higher than

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those set by an integrated monopoly. However, the complementarity relationship in the computing industry is quite different from that analysed by Cournot and others. For instance, an advanced application program enhances the value of an operating system (O/S), but it is useless without the O/S. In contrast, the O/S provides its basic functions without the advanced application program.

Furthermore, as Bresnahan (1999) and Bresnahan and Greenstein (1999) point out, in order to obtain a larger share of industry profits, a firm producing one product has an incentive to enter the others' "turf" by incorporating functions provided by the other firms. For example, in its early days, MS Windows did not include program functions such as WordPad, Internet Explorer (I.E.), and Windows Media, but over time it has included these and other programs that were previously supplied by independent firms. Another example is secondary cache. Once a separate piece of hardware, secondary cache is now integrated into the Intel CPU. As firms constantly try to expand their product boundaries, the boundaries between adjacent layers and the relationships among those products change continuously as a consequence of both vertical competition and technological innovation.

This paper analyses the strategic interactions between two firms whose products are asymmetrically complementary and attempts to shed light on vertical competition among different layers of the computing industry by exploring the effects of changes in their product boundaries.

To model asymmetric complementarity, we assume that the "main product" A, produced by firm A, by itself provides a performance quality of z, but consumers may derive a higher performance quality of q (i.e., $q > z$) by combining it with an "enhancer" product B, produced independently by firm B. Unlike the main product A, product B does not provide any function by itself.

To explore the implications of asymmetric complementarities between products A and B, we first analyse a simultaneous pricing game between firms A and B given z, $0< z <q$. It turns out that asymmetric complementarity combined with heterogeneous consumer preference over performance gives rise to the following three unexpected results. First, if z is relatively small, then products A and B are as if they are symmetrically complementary with $z = 0$ and are always sold as a bundle. Second, if z is sufficiently large, then firms A and B price their products as if they are independent, in which case some consumers buy A alone while others buy both products. This result has an implication on the "double mark-up" problem: Even though products A and B are asymmetrically complementary, the firms set their prices independently, and the "double

mark-up" problem vanishes. Third, over a certain range of intermediate value *z*, no pure-strategy Nash equilibrium exists. However, we can construct a mixed strategy equilibrium over the range.

Also, we examine the effects of increasing *z*, which can be interpreted as an expansion of firm A's product boundary. We analyse how an increase in *z* affects social welfare, industry profits and consumer welfare.

There are several recent related studies on complementary technologies and patents (e.g., Farrell and Katz 2000 and Lerner and Tirole 2002) and tying/bundling (e.g., Whinston (1990), Choi and Stefanadis (2001), Carlton and Waldman (2002), and Nalebuff (2004)).

Farrell and Katz (2000) analyze the incentive of a monopolist in product A to enter complementary product B's market in order to force independent suppliers of B to charge lower prices, which increases its own profits made from product A. If consumers in our model were homogeneous, then our results would become very similar to those of Farrell and Katz (2000): an increase in *z* "price squeezes" product B and always has a positive effects on firm A's profits. With heterogeneous consumer preference, however, we show that an increase in *z* does not have monotonic effects on firms' pricing and profits.

Our model is also closely related to Lerner and Tirole (2004)'s model of patent portfolios, which allows a full range of complementarity and substitutability. There are several major differences between our model and theirs. First, their focus is on factors that encourage or hinder the formation of patent pools and the welfare effect of these pools, whereas our focus is on the firms' switching pricing behavior and the welfare effects of changes in *z*. Second, in their model all users or licensees derive the same amount of marginal benefits from an additional patent, but in our model different consumer types derive different marginal benefits from the basic product A and the bundle (A+B). Because of these differences, we obtain the result that the demand for A and B is independent of each other if *z* is sufficiently large and that no pure-strategy equilibrium exists for intermediate values of *z*.

Our paper is related to the literature on tying/bundling because product A in our model can be regarded as a bundle of two complementary products, A_1 and B_1 (i.e., A_1 and B_1 combine to yield a performance quality *z*, whereas A_1 and B combine to yield a performance quality q.) However, this literature either focuses on the entry deterrence role of tying or assumes that tying with a firm's own product excludes consumption of competing products. [1] However, when

[1] In the literature on tying, if an incumbent ties its products, it is often assumed that consumers cannot untie the tied product, or consumers do not have any incentive to add an entrant's product

Microsoft ties its Windows O/S and its applications such as I.E., it still leaves room for consumers to add a rival product to its OS. We capture this product relationship by assuming that product B as an enhancer of the basic product A.

Nalebuff (2004) shows that when consumers are heterogenous in their valuations of products A and B, an incumbent, by bundling A and B, can significantly lower the profits of a single-product entrant and that bundling could be quite an effective entry deterrence strategy.[2] However, our paper looks at the case in which one firm produces only a base product, and the other firm produces a complementary product.

Section 2 develops a simple model, and section 3 analyzes the game and demonstrates the possible non-existence of pure-strategy Nash equilibrium. Section 4 analyzes the effect of z on firms' profits, consumer surplus, and social welfare. In section 5, we check the robustness of the main results when consumers' preferences vary along two dimensions. Concluding remarks follow in the final section.

2. A Model of product boundary

There are two firms, A and B, that provide complementary products A and B, respectively. Product A provides some basic functions, and its performance level is measured by a parameter z. Product B by itself does not provide any function, but enhances product A's performance. The combination of products A and B (denoted by (A+B) hereafter) provides a higher performance level $q \geq z$. Let product i's (i=A, B) price and unit production cost be denoted by p_i and c_i, respectively. We assume that the two firms set their prices simultaneously.

Given p_A and p_B, consumers make their purchase decisions. Consumers differ in their valuation of product quality. The utility function of a type-θ consumer, $\theta \in [0,1]$, is given by $\theta Q + I$, where I is her income spent on numeraire goods, and Q is a quality index of a product. Let the cumulative distribution function and continuous density functions be given by $G(\theta)$ and $g(\theta)$, respectively. Define $F(\theta)$ as the proportion of consumers whose type is higher than θ and $f(\theta)$ as F's density function, i.e., $F(\theta)=1-G(\theta)$, and $f(\theta)= -g(\theta)<0$. We make the standard assumption that the distribution of θ satisfies the increasing hazard rate condition:

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to the tied products because there is no quality difference between the incumbent and the entrant's products.

[2] Interestingly, the bundling decision increases the incumbent's profit even after entry occurs in one of these markets. In Whinston (1990), and Choi and Stefanadis (2001), tying is not a profitable choice for an incumbent if entry has already occurred.

namely, $-f(\theta)/F(\theta)$ is increasing in θ.[3] This increasing hazard rate condition yields strictly quasi-concave profit functions for firms A and B.

We impose the following restrictions on the model's key parameters throughout our analysis,

Assumption 1: $c_A + c_B \leq q$

$$0 \leq z \leq \bar{z} = q - c_B$$

The first restriction implies that the maximum willingness to pay for product (A+B) is larger than or equal to its unit production cost. Without this restriction, (A+B) will never be supplied. The second restriction implies that the quality enhancement brought about by product B (i.e., q-z) is larger than or equal to c_B. Without the second restriction, there will be no supply of product B. Under Assumption 1, both firms A and B are active and the classic double mark-up problem may arise.

Demand functions for products A and B

Consumer θ has three options: (i) to buy product A alone and gain net utility $V_A(\theta) = z\theta - p_A$; (ii) to buy (A+B) and gain net utility $V_{A+B}(\theta) = q\theta - p_A - p_B$; (iii) and to buy neither and gain zero net utility. A necessary condition for the consumer to buy A alone is $\theta \geq \theta_A = p_A/z$. Similarly, a necessary condition for a consumer to buy (A+B) is $\theta \geq \theta_{A+B} = (p_A + p_B)/q$. Consumers get additional benefits of $(q-z)\theta$ by purchasing product B in addition to product A. Thus, a necessary condition for a consumer to buy B in addition to A is that $\theta \geq \theta_B = p_B/(q-z)$.

Since $V_A(\theta)$ intersects the steeper function $V_{A+B}(\theta)$ at one point, θ_B, there are three possible cases.

Case 1. Virtually Independent Products: $\theta_A < \theta_{A+B} < \theta_B$.

This case is illustrated in Figure 1. Consumer types between θ_A and θ_B will buy product A alone, whereas consumer types $\theta \geq \theta_B$ will buy (A+B). That is, consumers with $\theta \geq \theta_A$ will

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[3] The increasing hazard rate condition is satisfied by most widely used distributions. See Fudenberg and Tirole (1991).

buy product A, and consumers with $\theta \geq \theta_B$ will additionally buy product B. Substituting the definition of θ_A and θ_B, the demand functions for A and B become:

$$D_A(p_A,p_B) = F(\theta_A) = F(\frac{p_A}{z}) \qquad (1)$$

$$D_B(p_A,p_B) = F(\theta_B) = F(\frac{p_B}{q-z})$$

As long as their prices satisfy $\theta_A < \theta_{A+B} < \theta_B$, the demand for A depends only on p_A, and the demand for B depends only on p_B. Firms A and B act as independent firms, and we call this case "virtually independent products" and refer to the firms' pricing as "independent pricing" in the rest of this paper. Let p^*_{A1} and p^*_{B1} denote the Nash Equilibrium prices under independent pricing. For example, when $F(\theta)=1-\theta$ (i.e., θ is uniformly distributed), the Nash equilibrium prices are $\frac{z+c_A}{2}$ and $\frac{q-z+c_B}{2}$, respectively.

Case 2: Virtually Strict Complements: $\theta_B < \theta_{A+B} < \theta_A$

Figure 2 illustrates this case. Consumers with $\theta < \theta_{A+B}$ will buy neither products, but consumers with $\theta \geq \theta_{A+B}$ will buy (A+B). None will buy product A alone. Substituting the definition of θ_{A+B}, the demand functions for A and B become:

$$D_A(p_A,p_B) = D_B(p_A,p_B) = F(\frac{p_A+p_B}{q}) \qquad (2)$$

As long as $\theta_B < \theta_{A+B} < \theta_A$, the demand for A and that for B depend on the total price ($p_A+ p_B$), exhibiting the characteristics of strict complements. Thus, in the rest of this paper we call this case "virtually strict complements" and refer to the firms' pricing of the virtually strict complements as "bundling pricing." Let $R_A(p_B)$ and $R_B(p_A)$ denote the firms' best response functions when the two faces demand system (2). We assume that $\frac{1}{|R'_A|} > |R'_B|$ so that there is a unique interaction of the two best response functions. The Nash equilibrium is denoted by (p^*_{A2} and p^*_{B2}). For example, when $F(\theta)=1-\theta$, we have $R_A(p_B)= \frac{q-p_B+c_A}{2}$, $R_B(p_A)= \frac{q-p_A+c_B}{2}$, and the Nash equilibrium is ($\frac{q-c_B+2c_A}{3}$, $\frac{q-c_A+2c_B}{3}$).

Case 3: Borderline between Virtually Strict Complements and Independent Products: $\theta_B = \theta_{A+B} = \theta_A$

Starting from this borderline case, the firms face demand system (1) either if firm A lowers its price or if firm B increases its price, however slightly. And they face demand system (2) either if firm A increases its price or if firm B lowers its price. In other words, the firms' demand functions meet at a kink where $\theta_B = \theta_{A+B} = \theta_A$. Lemma 2 in Section 3 says that the case $\theta_B = \theta_{A+B} = \theta_A$ cannot be a Nash equilibrium. An immediate implication is that if a pure-strategy Nash equilibrium exists, then the realized demand system must be either (1) or (2). Therefore (p^*_{A1} and p^*_{B1}) and (p^*_{A2} and p^*_{B2}) are the only candidates for a pure-strategy Nash equilibrium.

3. Analysis

Firm B's optimal pricing

We first examine firm B's optimal price given p_A. The demand function faced by firm B depends on the relative size of θ_B and θ_A.

$$D_B(p_A, p_B) = \begin{cases} D_{B1} = F\left(\frac{p_B}{q-z}\right) & \text{if } p_B \geq \frac{q-z}{z} p_A \\ D_{B2} = F\left(\frac{p_A+p_B}{q}\right) & \text{if } p_B \leq \frac{q-z}{z} p_A \end{cases}$$

Let us define the profit functions corresponding to D_{B1} and D_{B2} as $\Pi_{B1}(p_B; z) = F\left(\frac{p_B}{q-z}\right)(p_B - c_B)$ and $\Pi_{B2}(p_B; p_A) = F\left(\frac{p_A+p_B}{q}\right)(p_B - c_B)$, respectively. Firm B maximizes $\Pi_{B1}(p_B; z)$ subject to the constraint $p_B \geq \frac{q-z}{z} p_A$ and $\Pi_{B2}(p_B; p_A)$ subject to the constraint $p_B \leq \frac{q-z}{z} p_A$. The two profit functions intersect at $p_B = \frac{q-z}{z} p_A$.[4] Let p^*_{B1} and $R_B(p_A)$ denote the unconstrained optimal prices of Π_{B1} and Π_{B2}, respectively. Lemma 1 shows that firm B's overall profit function has a single peak for any given p_A and that its optimal price is unique.

[4] When $p_B = \frac{q-z}{z} p_A$, we have $\frac{p_A+p_B}{q} = \frac{1}{q}\left(\frac{z}{q-z} p_B + p_B\right) = \frac{1}{q-z} p_B$.

Lemma 1: *Firm B's optimal price depends on p_A and is continuous in p_A. There exist* $\underline{p}_A$ and $\overline{p}_A$, where $0 < \underline{p}_A < \overline{p}_A < q$-$c_B$, such that

$$p_B^* = \begin{cases} p_{B1}^* & \text{if } p_A \leq \underline{p}_A \\ \frac{q-z}{z} p_A & \text{if } \underline{p}_A \leq p_A \leq \overline{p}_A \\ R_B(p_A) & \text{if } \overline{p}_A \leq p_A \leq q - c_B. \end{cases}$$ [5]

Proof. See the Appendix.

If p_A is zero, then clearly all consumers will get product A, and the only question is who will buy product B additionally. From the point of view of firm B, it faces demand system (1) and maximizes its profit along D_{B1} by setting its optimal price, p_{B1}^*, or equivalently by selling its products to consumers whose types are above the cut-off point $\theta_B^* = p_{B1}^*/(q\text{-}z)$. As p_A increases, fewer consumers will buy product A, but as long as the lowest consumer type that buys product A is lower than $p_{B1}^*/(q\text{-}z)$, firm B's optimal price remains unconstrained by p_A.

However, once p_A exceeds the threshold $\underline{p}_A$ but remains below $\overline{p}_A$, then the constraint becomes just binding, so firm B's optimal price occurs at the kink $\frac{q-z}{z} p_A$.

If p_A is higher than $\overline{p}_A$, then no consumer is interested in buying product A alone, and products A and B are always sold together as a bundle, so B's optimal price becomes $R_B(p_A)$.

Figure 3 illustrates how firm B's optimal price responds to p_A in the case of a uniform distribution.[6] As the figure shows, firm B's best response is not monotonic.

Firm A's optimal pricing

Similar to firm B's case, firm A's demand curve consists of two connected segments D_{A1} and D_{A2}. When p_B is taken as given, firm A maximizes $\Pi_{A1} = F(\frac{p_A}{z})(p_A\text{-}c_A)$ subject to $p_A \leq \frac{z}{q-z} p_B$, but maximizes $\Pi_{A2} = F(\frac{p_A+p_B}{q})(p_A\text{-}c_A)$ subject to $p_A \geq \frac{z}{q-z} p_B$. Let p_{A1}^* and $R_A(p_B)$ denote the

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[5] If $p_A > q\text{-}c_B$, then $p_B^* = c_B$, and none will buy product B.

[6] In this case, $\underline{p}_A = \frac{z(q-z+c_B)}{2(q-z)}$, $\overline{p}_A = \frac{z(q+c_B)}{2q-z}$, $p_{B1}^* = \frac{q-z+c_B}{2}$, and $R_B(p_A) = \frac{q-p_A+c_B}{2}$.

unconstrained optimal p_A for the profit functions, Π_{A1} and Π_{A2}, respectively. The following lemma describes firm A's optimal price.

Lemma 2 There exist $\tilde{p}_B$ such that if $p_B \leq \tilde{p}_B$, then firm A sets its price equal to $R_A(p_B)$, and we have $\theta_A^* > \theta_B^*$; if $p_B \geq \tilde{p}_B$, firm A sets its price at p_{A1}^*, and we have $\theta_A^* < \theta_B^*$. Also, since $R_A(\tilde{p}_B) > p_{A1}^*$, firm A's optimal price is not continuous in p_B at $p_B = \tilde{p}_B$.

Proof. See the Appendix.

While products A and B are asymmetric complements, products A and (A+B) are substitutes for each other. Firm A chooses between two different pricing strategies: "independent pricing" (i.e., selling product A as a stand-alone product) or "bundling pricing" (i.e., selling it as a part of the bundle (A+B)). That is, firm A can sell its product as a stand-alone low-quality product or can sell it as a component of a high-quality product. If p_B is sufficiently low, then firm A will find it profitable to choose bundling pricing. In contrast, if p_B is relatively high, then the demand for (A+B) is limited by the high price of product B, so firm A may find it more profitable to choose independent pricing.

These considerations behind firm A's optimal pricing strategies are quite similar to those discovered by Gabszewicz and Wauthy (2003).[7] In Gabszewicz and Wauthy's model, there are two firms supplying vertically differentiated stand-alone products, but consumers also have an option of "joint purchase" of both products. They analyse how the joint purchase affects price competition between duopolists under the assumption of uniform distribution and zero production costs. They find that "a firm faces two different pricing strategies: either it charges relatively low prices and fights for market shares or it "retreats" with high price on the "rich" side of the market where "joint purchasers" are located."

Nash equilibrium

[7] In Gabszewicz and Wauthy's model, products have their own stand-alone values. However, in our model product B is useless by itself. Given this difference, in our model we have an equilibrium in which the two firms behave as independent firms. Such an equilibrium cannot arise in Gabszewicz and Wauthy's model.

Lemma 2 has an implication for finding Nash equilibria. If p_B is lower than $\tilde{p}_B$, then firm A sets its price equal to $R_A(p_B)$, which is strictly larger than $\frac{z}{q-z}p_B$, and we have $\theta_A^* > \theta_B^*$; If p_B is higher than $\tilde{p}_B$, firm A sets its price equal to p^*_{A1}, which is strictly less than $\frac{z}{q-z}p_B$, and we have $\theta_A^* < \theta_B^*$. So the lemma implies that $\theta_A^* = \theta_B^*$ cannot be a Nash equilibrium. Therefore, if a pure-strategy Nash equilibrium exists, then in equilibrium either both firms adopt independent pricing ($\theta_A < \theta_B$) or both firms adopt bundling pricing ($\theta_B < \theta_A$). From this result Corollary 1 follows.

*Corollary 1***.** The prices (p^*_{A1}, p^*_{B1}) and (p^*_{A2}, p^*_{B2}) are the only candidates for a pure-strategy Nash equilibrium. The former outcome results from both firms adopting independent pricing, while the latter outcome results from both firms adopting bundling pricing.

An example of non-existence of pure-Strategy Nash equilibrium[8]

Even though firm B's optimal price is everywhere continuous in p_A, the existence of a pure-strategy Nash equilibrium cannot be guaranteed because firm A's optimal price is not continuous at $\tilde{p}_B$. The following simple example provides an illustration.

Assume that the production costs c_A and c_B are zero, q=1, and θ is uniformly distributed between zero and one. From Corollary 1 we need to consider only (p^*_{A1}, p^*_{B1}) and (p^*_{A2}, p^*_{B2}) as candidates for a pure-strategy Nash equilibrium.

We first verify whether (p^*_{A1}, p^*_{B1}), resulting from independent pricing, is a Nash equilibrium. A necessary condition for these prices to be a Nash equilibrium is that the two firms face demand system (1). Given this demand system and zero production costs, firm A maximizes $(1-\frac{p_A}{z})p_A$, and firm B maximizes $(1-\frac{p_B}{1-z})p_B$. Firm A's first-order condition is given by

$$(1-\frac{p_A}{z})-\frac{p_A}{z} = 0, \qquad (3)$$

[8] We would like to thank an anonymous referee for providing this example and the intuition behind it.

and firm B's first-order condition is given by

$$(1-\tfrac{p_B}{1-z})\tfrac{p_B}{1-z} = 0 \quad (4)$$

The solution of the two first-order conditions (3) and (4) is given by ($p^*_{A1} = \frac{z}{2}$, $p^*_{B1} = \frac{1-z}{2}$), which yields $\theta_A = \theta_B = \frac{1}{2}$, implying that both firms target the same marginal consumer. But the discussion leading to Corollary 1 shows that $\theta_A = \theta_B$ cannot be a Nash equilibrium because firm A has an incentive to deviate from $\theta_A = \theta_B$. A non-technical explanation is as follows.

The first-order condition (3) says that if firm A decreases (increases) its price by d, its consumers increase (decrease) by $\frac{d}{z}$. However, when firm A increases its price, the impact of this price change on firm A's profit is different from that captured by the first order condition (3).[9] If firm A does increase its price by d, then $\theta_A > \theta_B$, and its marginal consumer is determined by θ_{A+B}. Thus its consumers decrease by d, which is less than $\frac{d}{z}$ as indicated by (3). That is, when p_A goes up, fewer customers walk away from product A than as indicated by (3) because some of those who decide not to buy A alone may buy the bundle (A+B) instead. Since the decrease in the sale of product A due to an increase in p_A is less than that indicated by the first-order condition (3), independent pricing cannot be a Nash equilibrium. As Lemma 2 indicates, it is firm A that wants to deviate from $\theta_A = \theta_B$.

With regard to the outcome (p^*_{A1}, p^*_{B1}), firm B's first-order condition (4) implies that its consumers increase by d/(1-*z*) if firm B decreases its price by d. It is not profitable for firm B to decrease its price from p^*_{B1} when it could attract d/(1-*z*) additional customers by lowering its price by d. Starting from $\theta_A = \theta_B$, if firm B lowers its price by d, then $\theta_A > \theta_B$ and its marginal consumer is determined by θ_{A+B}, implying that it will only gain d customers. Thus, attracting even

[9] If firm A decreases its price, the impact on its profit is the same as indicated by (3): if firm A does decrease its price by d, then we have $\theta_A < \theta_B$, and its marginal consumer is determined by θ_A. Thus its consumers increase by $\frac{d}{z}$, as indicated by (3).

fewer customers by lowering its price makes it strictly unprofitable. Thus, firm B's optimal price is $p^*_{B1} = \frac{1-z}{2}$ if firm A sets $p^*_{A1} = \frac{z}{2}$.[10]

Second, let us check whether bundling pricing (p^*_{A2}, p^*_{B2}) is a Nash equilibrium. A necessary condition for these prices to be a Nash equilibrium is that the two firms face demand system (2). At the candidate Nash equilibrium ($p^*_{A2} = \frac{1}{3}$, $p^*_{B2} = \frac{1}{3}$), each firm earns $\frac{1}{9}$. Since product A provides quality level z by itself, firm A can always sell its product at least half of the consumers by setting p_A equal to $\frac{z}{2}$, yielding a profit of $\frac{z}{4}$. Thus, if z is larger than $\frac{4}{9}$, then the symmetric price vector ($p^*_{A2} = \frac{1}{3}$, $p^*_{B2} = \frac{1}{3}$) cannot be a Nash equilibrium because firm A can do better by switching to independent pricing ($p_A = \frac{z}{2}$). In contrast, for $z \le \frac{4}{9}$, firm A does not have any incentive to deviate from bundling pricing to independent pricing. Since firm B does not have any incentive to deviate from $p^*_{B2} = \frac{1}{3}$ given $p^*_{A2} = \frac{1}{3}$,[11] the symmetric price vector ($p^*_{A2} = \frac{1}{3}$, $p^*_{B2} = \frac{1}{3}$) is a Nash equilibrium if $z \le \frac{4}{9}$.

In summary, if $z \le \frac{4}{9}$, then ($p^*_{A2} = \frac{1}{3}$, $p^*_{B2} = \frac{1}{3}$) is a Nash equilibrium, which is identical to that for the case z= 0, the case of strict complements. There is no pure-strategy Nash equilibrium if $\frac{4}{9} < z < 1$.[12]

In this example the independent pricing cannot be a Nash equilibrium when the two firms target the same marginal consumer. The intuition is extendable to the general cases with positive c_A and c_B and general distribution function F(θ). Under the independent pricing, firm A maximizes $zF(\theta_A)(\theta_A - \frac{c_A}{z})$, and firm B maximizes $(q-z)F(\theta_B)(\theta_B - \frac{c_B}{q-z})$, respectively. The firms' optimal cut-off points θ_A and θ_B depend on $\frac{c_A}{z}$ and $\frac{c_B}{q-z}$, respectively. When $\frac{c_A}{z} = \frac{c_B}{q-z}$, independent pricing yields $\theta_A = \theta_B$, which cannot be a Nash equilibrium. The reason is the same: if firm A's marginal customer is already buying B, then a price increase by firm A will not lead to as large a loss in customers as would be the case if firm A were acting without product B. This

[10] Starting from $\theta_A = \theta_B$, if firm B raises its price by d, then $\theta_A < \theta_B$ and it will lose d/(1-z) customers as indicated by (4)

[11] With regard to the outcome (p^*_{A2}, p^*_{B2}), a general proof of Proposition 1 shows that it is optimal for firm B to set its price at p^*_{B2} if firm A chooses p^*_{A2}.

[12] If z=1, there is a Nash equilibrium in which firm B sets its price at zero, and firm A behaves as a single monopolist.

gives firm A an incentive to raise its price, thus destroying independent pricing as a pure-strategy Nash equilibrium. By continuity, the same argument goes through if $\frac{c_A}{z}$ is sufficiently close to $\frac{c_B}{q-z}$ (or equivalently, if firm A's market share is sufficiently close to firm B's market share).

General cases

In this section we characterize pure-strategy Nash equilibria for the general cases of our model. We will show that such equilibria exist if z is either sufficiently small or if z is sufficiently large. However, there is an intermediate range of z within which no pure strategy equilibrium exists, even though a mixed strategy equilibrium exists.

To understand intuitively the general non-existence result, we apply the logic used in the above example. Consider one of the Nash equilibrium candidates under bundling pricing, (p^*_{A2} and p^*_{B2}). In this case firm A earns Π^*_{A2}, which are independent of z. If firm A chooses to deviate from bundling pricing by setting its price at p^*_{A1}, it earns at least $\Pi^*_{A1} = F(\frac{p^*_{A1}}{z})(p^*_{A1} - c_A)$, which is monotonically increasing in z. Since Π^*_{A2} is independent of z, there exists a unique z_{min} such that $\Pi^*_{A1} > \Pi^*_{A2}$ if and only if $z > z_{min}$ Thus, if $z > z_{min}$, then firm A will choose independent pricing, implying that (p^*_{A2} and p^*_{B2}) cannot be a Nash equilibrium; if $z \le z_{min}$, then firm A would not deviate from bundled sales, and (p^*_{A2} and p^*_{B2}) becomes a Nash equilibrium.

Similarly, consider the other Nash equilibrium candidate under independent pricing, (p^*_{A1} and p^*_{B1}), in which case firm A makes Π^*_{A1}. However, if firm A switches to bundling pricing, its optimal price is $R_A(p^*_{B1})$, and its resultant profit is $T^*_2(z) = F(\frac{R_A(p^*_{B1})+p^*_{B1}}{q})(R_A(p^*_{B1}) - c_A)$. Let $T(z)$ be defined as $T(z) = \Pi^*_{A1}(z) - T^*_2(z)$, which is continuous in z. It can be shown that $T(z)>0$ if z is sufficiently large and that $T(z)<0$ if z is sufficiently small, which implies that, if z is large enough, independent pricing is a Nash equilibrium and that, if z is sufficiently small, independent pricing cannot be a Nash equilibrium. Since $T(z)$ is continuous in z, there is at least one z such that $T(z)=0$. Let the largest solution of $T(z) = 0$ be denoted by z_{max} . It follows that $T(z) > 0$ if $z_{max} < z$. That is, for $z \ge z_{max}$, (p^*_{A1} and p^*_{B1}) is a Nash equilibrium.

The only question left is what happens between z_{min} and z_{max}. If $T(z) = 0$ has multiple (always an odd number of) solutions, then there are alternating sub-intervals over which $T(z) > 0$

and sub-intervals over which $T(z)<0$. For instance, consider the case of three solutions.[13] Let the three solutions be given by z_1, z_2 and $z_{\max}$. Then, as figure 4 shows, $T(z)<0$ for $z < z_1$; $T(z)>0$ for $z_1 < z < z_2$; $T(z) < 0$ for $z_2 < z < z_{\max}$, and $T(z) > 0$ for $z > z_{\max}$. It implies that for $z_{\min} < z < z_1$ there is no pure-strategy Nash equilibrium; for $z_1 < z < z_2$, independent pricing is a pure-strategy Nash equilibrium; for $z_2 < z < z_{\max}$ there is no pure-strategy Nash equilibrium; for $z > z_{\max}$, independent pricing is a pure-strategy Nash equilibrium.

If $T(z) = 0$ has only one solution, then the unique solution is $z_{\max}$, and $T(z)<0$ for the interval $(z_{\min}, z_{\max})$, indicating that a pure-strategy equilibrium does not exist over the entire interval. A sufficient condition for the uniqueness is that $T(z)$ monotonically increases in z. Since $T(z)=\Pi^*_{A1}(z) - T_2^*(z)$, by the envelop theorem, we have $\frac{\partial T(z)}{\partial z} = \frac{\partial \Pi^*_{A1}}{\partial z} - \frac{\partial \Pi^*_{A2}}{\partial p^*_{B1}} \frac{\partial p^*_{B1}}{\partial z}$. If the direct effect of z on Π^*_{A1} is larger than the effect of z on Π^*_{A2} through p^*_{B1}, then $T(z)$ is monotonically increasing in z and $T(z)=0$ has a unique solution.[14]

For those intervals over which no pure-strategy equilibrium exists, we can construct a mixed-strategy equilibrium. The following proposition characterizes Nash equilibrium for different values of z.

Proposition 1: There exist two unique critical values of z: $z_{\min}$, $z_{\max}$, with $0 < z_{\min} < z_{\max} \leq q-c_B$ such that the following holds:

(a) If $z \leq z_{\min}$, there exists a unique pure-strategy Nash equilibrium with bundling pricing. The outcome is identical to the equilibrium for strict complements, i.e., $z=0$, as products A and B are always consumed as a bundle (A+B).

(b) If $z \geq z_{\max}$, there exists a unique pure-strategy Nash equilibrium with independent pricing. Some consumers buy A alone while other consumers buy (A+B).

(c) If $T(z) = 0$ has only one solution, then no pure-strategy equilibrium exists over the entire interval $(z_{\min}, z_{\max})$. If $T(z)=0$ has multiple solutions, then over $(z_{\min}, z_{\max})$, there exist alternating sub-intervals such that over some intervals there exists a pure-strategy Nash equilibrium with independent pricing, and over the remaining intervals no pure-strategy Nash equilibrium exists.

(d) When there is no pure-strategy equilibrium, we can construct a mixed-strategy equilibrium. The mixed-strategy equilibrium is characterized by a quadruplet [α, p^*_{A1},

[13] The arguments can be generalized to any odd number of solutions.

[14] Notice that monotonicity is a sufficient, but not a necessary condition for uniqueness. This condition is satisfied for uniform distribution $F(\theta)$.

$R_A(\tilde{p}_B)$, $\tilde{p}_B$], where firm B sets its price at $\tilde{p}_B$, α is the probability of firm A's adopting p^*_{A1}, and (1- α) is the probability of firm A's adopting $R_A(\tilde{p}_B)$.

Proof. See the Appendix.

Proposition 1 says that if z is small, then bundling pricing is a Nash Equilibrium. After z increases beyond a certain point, firm A would rather set its price to sell product A alone without consideration of product B. This independent pricing by firm A, however, cannot be part of a pure-strategy Nash Equilibrium, if Firm B targets at roughly the same set of consumers. The converse of this result is that independent pricing can be a pure-strategy Nash equilibrium only if firm A's market share is much larger than that of firm B.[15] As an example, suppose that q=1, z=0.7, c_B = 0.1 and c_A = 0. Independent pricing leads to $\theta_A = 0.5$, and $\theta_B = 0.67$, in which firm A's market share is much larger than that of firm B. These prices constitute a pure-strategy Nash equilibrium because firm A will lose customers rapidly if it raises its price above p^*_{A1} . No customers will continue to buy A as part of (A+B) if they decide not to buy A alone, unless the price is raised all the way to the point at which θ_A becomes equal to θ_B.

The existence and non-existence of a pure-strategy Nash equilibrium are illustrated in Figures 5. If $z > z_{max}$, as illustrated in Figure 5-(a), in equilibrium the firms adopt independent pricing; if $z < z_{min}$, as illustrated in Figure 5-(c), in equilibrium the firms adopt bundling pricing. The nature of a mixed-strategy Nash equilibrium is illustrated in Figure 5-(b). If firm B sets its price at $\tilde{p}_B$, as Lemma 2 shows, firm A is indifferent between p^*_{A1} and $R_A(\tilde{p}_B)$ and is therefore indifferent between any randomization of them. There exist probabilities α for p^*_{A1} and (1-α) for $R_A(\tilde{p}_B)$ such that firm B's best response to firm A's randomization is exactly $\tilde{p}_B$.

If there is only one type of consumer, firm A, by virtue of the fact that product A is essential to the enjoyment of product B, could extract some of the additional value created by product B via aggressive pricing, i.e., firm A sets a higher price for its product than in the absence

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[15] Under independent pricing, firm A's market share (*resp*. firm B's market share) depend on $\frac{c_A}{z}$ (*resp*. $\frac{c_B}{q-z}$). Thus, it happens when z is much larger than $\frac{qc_A}{c_A+c_B}$.

of product B, taking out some values created by firm B.[16] Proposition 1, however, raises questions about the validity of this idea. Part (b) says that over some range of *z* there is no pure -strategy Nash equilibrium. And when such an equilibrium exists, part (c) says that if the extent of the quality enhancement, (q-*z*), is small, then product B does not affect firm A's pricing and profits.

4. The impact of *z* on firm profits, consumer surplus, and social welfare

Quite often a dominant firm in one layer extends its product boundary to include functions that are traditionally provided by its complementors. In this section, we analyse the impact of *z* on the firms' prices and profits, consumer surplus, and social welfare for the two basic cases identified in Proposition 1 (namely, bundling pricing and independent pricing). In the analysis, we assume that c_A does not change with *z*, which is a good approximation for the software industry.[17] If c_A is allowed to increase with *z*, our main comparative statics results in the independent pricing case still hold so long as $\frac{z}{c_A(z)}$ is increasing in *z*, i.e., the production cost per unit of *z* is a decreasing function of *z*.[18]

Virtually strict complements under bundling pricing

[16] When there is only a single consumer type, the existence of product B will lead to a higher price for product A. As an example, suppose the consumer values product A at 100 and (A+B) at 110. Then any price (p_A= x, p_B =110-x) is a Nash equilibrium as long as 110≥ x ≥100. In Carlton and Waldman (2002), for instance, the surplus created by the complementary good is assumed to be equally divided between the entrant and the incumbent firm.

[17] For instance, even though Microsoft expands the functions of Windows by incurring R&D costs, the marginal production cost of a copy of the Windows would remain roughly the same as before.

[18] In the case of virtually strict complements, the results are different from those reported in Proposition 2. See the next footnote

If z is small, in equilibrium the firms adopt bundling pricing, and consumers buy only the bundle (A+B). The firms' equilibrium prices are identical to those when $z = 0$,[19] and there is a double mark-up problem. Thus, we have

Proposition 2: If $z \in [0, z_{\min})$, then the equilibrium prices are the same as those for $z = 0$. Consumers always buy A and B as a bundle, and the double mark-up problem persists. The firms' profits, consumer surplus and social welfare are independent of z.

Virtually independent products under independent pricing

Intuitively, one expects that more consumers buy A and fewer consumers buy B as A's own performance z improves. That is indeed the case when A and B are virtually independent, i.e. $z \geq z_{\max}$.

Under independent pricing, firm A maximizes $\Pi_A = F(\theta_A)(p_A\text{-}c_A) = zF(\theta_A)(\theta_A\text{-}\frac{c_A}{z})$ by choosing θ_A^*. Firm A's optimal cut-off point θ_A^* is an increasing function of $\frac{c_A}{z}$, which implies that as z increases more consumers buy product A. Similarly, firm B choose θ_B^* to maximize $\Pi_B = F(\theta_B)(p_B\text{-}c_B) = (q\text{-}z)F(\theta_B)(\theta_B\text{-}\frac{c_B}{q-z})$. Firm B's optimal cut-off θ_B* is an increasing function of $\frac{c_B}{1-z}$, which implies that as z increases fewer consumers buy product B.

To better understand the classical double mark-up problem for the case of independent pricing, we need to characterize the optimal pricing strategies of an integrated monopoly that sells both products A and B. It maximizes Π^{Int} as defined in (5) by choosing two optimal cut-off points, x_A and x_B, where types higher than x_A buy product A, and types higher than x_B buy product B additionally. Since consumers will not buy B without buying A, the firm's choice variables are subject to the constraint $x_A \leq x_B$.

$$\Pi^{\text{Int}} = F(x_A)(p_A\text{-}c_A) + F(x_B)(p_B\text{-}c_B)$$
$$= zF(x_A)(x_A\text{-}\frac{c_A}{z}) + (q\text{-}z)F(x_B)(x_B - \frac{c_B}{q-z}) \quad (5)$$

[19] Under bundled pricing, if c_A increases in z, then z affects the firms' prices only through c_A. Thus, the effects of increasing z are the same as those of increasing c_A. That is, as z increases, p_A^* goes up but p_B^*, social welfare as well as the two firms' profits go down.

Obviously, the profit function Π^{Int} is equal to the sum of Π^A and Π^B. In the absence of the constraint $x_A \leq x_B$, these two parts of (5) can be maximized independently.

If $\frac{c_A}{z} \leq \frac{c_B}{q-z}$, or equivalently if $z \geq \frac{q\,c_A}{c_A+c_B}$, then independent maximization leads to $x_B^* \geq x_A^*$, i.e., the constraint is automatically satisfied, and the integrated firm can ignore the constraint. In contrast, if $\frac{c_A}{z} > \frac{c_B}{q-z}$, or equivalently if $z < \frac{qc_A}{c_A+c_B}$, then the constraint becomes binding, implying that the integrated firm will set $x_B^* = x_A^*$. That is, if $z \geq \frac{q\,c_A}{c_A+c_B}$, the integrated firm sells the two products as if the two products are independent of each other, and if $z < \frac{qc_A}{c_A+c_B}$, the firm sells the two products as if they are strict complements.

Since $z_{max} > \frac{q\,c_A}{c_A+c_B}$, if $z \geq z_{max}$, then the two firms A and B behave independently, and the prices set by firms A and B are equal to those set by the integrated monopoly. That is, the double mark-up problem disappears completely. Also, since the integrated firm's profit increases in *z*, the industry profits (the sum of the two firms' profits) increase in *z*.

Let us investigate the effect of an increase in *z* on the consumer surplus. In the case of virtually independent products, as *z* increases, θ_B increases but θ_A decreases. The latter implies that some new consumers buy product A, which adds to total consumer surplus. The former implies that some consumers switch from buying the bundle (A+B) to buying product A alone.

If θ is uniformly distributed, then in equilibrium the total price ($p_{A1}^* + p_{B1}^*$) is $\frac{q+c_A+c_B}{2}$, which does not change with *z*. Given that the price of the bundle is unaffected by *z*, by revealed preference consumers who switch from the bundle to product A alone are better off, whereas consumers who stay with the bundle are neither better off nor worse off. Thus, considering the impact on new consumers who buy A alone, consumers who continue to buy the bundle, and consumers who switch from the bundle to A alone, aggregate consumer surplus is definitely larger when *z* increases.

Proposition 3: Suppose the products are virtually independent as firms adopt independent pricing ($z \geq z_{max}$). Then, when c_A and c_B are positive , as *z* increases:

(a) p_{A1}^* increases and p_{B1}^* decreases;

(b) more consumers buy product A, and fewer consumers buy product B;

(c) firm A's profits increase, firm B's profits decrease, and the industry profits (the sum of the two firms' profits) increase. Also, there is no double mark-up problem.
(d) When θ is uniformly distributed, the total consumer surplus as well as the industry profits increase.

However, for a general distribution, G(θ), it is not clear how ($p^*_{A1} + p^*_{B1}$) changes with *z*. If the total price goes up, then those consumers who buy the bundle both before and after *z* increases will lose. To ascertain such a possibility, let us consider the following example in which, as *z* increases from some given value, not only the aggregate consumer surplus, but also social welfare decline. Suppose the distribution function of θ is given by $G(\theta)= \theta^2$, one that satisfies the standard increasing hazard rate condition. Suppose further that q=1, c_A=0.2 and c_B=0.3. It can be verified that z_{min}=0.41 and z_{mzx}=0.45, so at *z*= 0.5 there exists a pure-strategy Nash equilibrium under independent pricing. It can be shown that (i) $\frac{\partial (p_A + p_B)}{\partial z} > 0$ and (ii) $\frac{\partial SW}{\partial z} < 0$ at *z* =0.5. Since the industry profits increase with *z* for all *z*, the latter result implies that aggregate consumer surplus must have decreased as *z* increases from *z*=0.5.

Mixed-strategy Nash equilibrium

In this section, we examine the properties of the mixed-strategy Nash equilibrium identified in Proposition 1, [α, p^*_{A1}, $R_A(\tilde{p}_B)$, $\tilde{p}_B$] for *z* between z_{min} and $z_{max.}$

From the definition of a mixed-strategy, given $\tilde{p}_B$ firm A is indifferent between p^*_{A1} and $R_A(\tilde{p}_B)$, i.e., $\Pi_{A1}(p^*_{A1}) = \Pi_{A2}(R_A(\tilde{p}_B))$. From Proposition 3, we already know that p^*_{A1} is an increasing function of *z*. Since firm A is indifferent between the two prices, its expected profit is equal to Π_{A1}, which increases with *z*. Thus, firm A's expected profit increases in *z*. Also, from the definition of the mixed-strategy Nash equilibrium, when *z* increases, Π_{A2} must also increase, which can occur only if $\tilde{p}_B$ decreases. Since $R_A(p_B)$ is a decreasing function of p_B, its follows that $R_A(\tilde{p}_B)$ increases in *z*. Because as *z* increases, both p^*_{A1} and $R_A(\tilde{p}_B)$ increase, implying that the expected price of product A is an increasing function of *z*. To sum up, we have

***Proposition 4*:** In the mixed-strategy Nash equilibrium, as *z* increases, the expected price of product A go up, the price of product B goes down, and firm A's expected profits go up.

The α in the mixed strategy is chosen so that firm B's optimal price to the randomization becomes $\tilde{p}_B$. Unfortunately, we are not able to show how firm B's profit and α change with *z* generally.

5. Extensions to two dimensions of consumer heterogeneity

In the basic model studied in Sections 2-4, a consumer of type θ derives satisfaction $z\theta$ from product A and $q\theta$ from (A+B), respectively, which implies that consumers' preferences toward these two products are perfectly correlated. As Figures 1 and 2 show, each firm's marginal consumer is uniquely determined either only by its own price or only by the sum of prices. As a result, each firm sets its price as if the two products were either independent or strict complements.[20] Under these conditions, we show that firm A's best response function is not continuous when firm A's optimal pricing scheme changes from bundling pricing to independent pricing, and therefore for some intermediate values *z* there is no pure-strategy Nash equilibrium.

To check the robustness of the results obtained in the previous sections, we now consider a more general demand specification. Suppose that consumers derive $U_A(\theta, v) = v + z\theta$ from the main product A and derive $U_{A+B}(\theta, v) = v + q\theta$ from the bundle (A+B). An interpretation is that product A provides two functions, and product B enhances the quality of A's second function from *z* to q. To make this new preference genuinely different from that of the basic model, we assume that consumer preferences vary along *v* as well as θ.[21] In the remainder of this section, we assume (θ, *v*) to be distributed over $[0, 1] \times [0, \bar{v}]$.[22] The vertical axis measures *v*, the consumers' valuation of the first function of product A, and the horizontal axis measures θ as before. A necessary condition for a consumer to buy product A alone is $V_A(\theta, v) = U_A(\theta, v) - p_A \geq 0$. Similarly, a necessary condition for a consumer to buy (A+B) is $V_{A+B}(\theta, v) = U_{A+B} - p_A - p_B \geq 0$.

[20] We would like to thank an anonymous referee for providing us with this insight and for making several useful suggestions about the analysis of more general preferences.

[21] If *v* is a constant for all consumers, then $V_A(\theta) = z\theta-(p_A- v)$ and $V_{A+B}(\theta) = q\theta-(p_A-v+p_B)$, and the analysis in Sections 2-4 applies to this new demand specification if p_A is replaced by (p_A-v). All of the earlier results remain intact.

[22] The distribution of v between $[0, \bar{v}]$ can be interpreted as if the quality level of the first function is $\bar{v}$, and the consumers' valuation of the quality is distributed over [0, 1].

Finally, a necessary condition for a consumer to buy the bundle (A+B) rather than A alone is $\theta \geq \theta_B = p_B/(q\text{-}z)$.

Demand functions for products A and B

For simplicity, we set q=1. As shown in Figures 6, the nature of the demand functions for A and B depends on whether the critical value of the intersection of V_A and V_{A+B} occurs above, below, or within $[0, \bar{v}]$.[23]

If $(p_A - \frac{z}{1-z} p_B) > \bar{v}$, the situation is as depicted in Figure 6 (a).[24] Consumers above V_{A+B} will buy the bundle (A+B). The demand for either product depends on the total price (p_A+p_B), so the two firms behave as if the two products are strict complements. In this case the value of z does not affect the firms' pricing and profits.

If $(p_A - \frac{z}{1-z} p_B) < 0$, the situation is as depicted in Figure 6 (b).[25] Consumers above V_A buy product A, and consumers to the right of θ_B buy the bundle (A+B). Demand for the two products becomes independent: the demand for B (the dark area) depends only on p_B, and the demand for A (light area + dark area) depends on p_A alone. Thus, in this case each product's demand depends only on its own price, and there is no double mark-up problem.

If $\bar{v} \geq (p_A - \frac{z}{1-z} p_B) \geq 0$, then the situation is as depicted in Figure 6 (c). Demand for B is given by the dark area, whereas demand for A is given by both the light and dark areas. In this case, the demand for product B is a function of (p_A+p_B) as well as p_B, and the demand for product A depends on (p_A+p_B) as well as p_A. In Sections 2-4 where consumers differ only in θ, each firm's marginal consumer is uniquely determined by either its own price alone or by (p_A+p_B) alone. However, in the two-dimensional case, this simple dichotomy is lost, because each firm may have a continuum of marginal consumers, which depends on both its own price and the total price. Therefore, we refer to the situation depicted in Figure 6(c) as an intermediate case of "mixed demand."

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[23] The intersection of V_A and V_{A+B} occurs at ($p_B/(1-z)$, $p_A - \frac{z}{1-z} p_B$). Thus, depending on whether ($p_A - \frac{z}{1-z} p_B$) is larger than or less than 0 and $\bar{v}$, the intersection of V_A and V_{A+B} occurs above, below, or within $[0, \bar{v}]$.

[24] The inequality implies that $p_A > \bar{v}$.

[25] The inequality implies that $\frac{p_A}{z} < \frac{p_B}{1-z}$. Also, since $\frac{p_B}{1-z}$ is less than 1, it implies that $\frac{p_A}{z} < 1$.

Nash equilibrium under uniform distribution

Let us check the existence of Nash equilibrium *numerically* under the assumption that θ and *v* are distributed independently and uniformly over [0, 1] x [0, $\bar{v}$].

In the previous one-dimensional case, firm A's optimal price is discontinuous in p_B when firm A's optimal pricing regime switches from independent pricing to bundling pricing. But in the two-dimensional case, if $\bar{v}$ is large enough, then there are enough consumers who are willing to buy product A alone. Thus, bundling pricing is not firm A's optimal pricing scheme even for a case in which *z* is low.

For instance, if $\bar{v}$ =1 and $c_A = c_B = 0$, then for any price p_B between zero and one,[26] bundling pricing cannot be firm A's optimal price for all values of *z*.: In this case, when *z* = 0, firm A's optimal price results in the mixed demand, and firm A's optimal price is continuous in p_B. Also, when *z*= 0.7, if p_B is high, firm A's optimal price results in independent pricing, and if p_B gets lower, firm A switches to the mixed demand regime, and firm A's optimal price is continuous in p_B when it switches from the independent pricing regime to the mixed demand regime.[27] As a result, firm A's optimal price is continuous in p_B, and a pure-strategy Nash equilibrium exists for all values of *z*.

However, if $\bar{v}$ is small enough, then firm A's optimal pricing scheme would be bundling pricing for some values of *z*, and there is discontinuity in firm A's best response to p_B when firm A switches from bundled sales to the mixed demand. Thus, for some values of *z*, there is no pure-strategy Nash equilibrium, which is similar to the results in Proposition 1. An example is *z*= 0.4, $\bar{v}$ =0.12 and $c_A = c_B = 0$. The intuition is as follows: If $\bar{v}$ = 0, then the two-dimensional case is reduced to the one-dimensional model, where the non-existence of pure-strategy Nash equilibrium has been established. If $\bar{v}$ is small, then there are not enough consumers who buy product A alone. Thus, for a small value of *z*, bundling pricing becomes attractive to firm A. In other words, small $\bar{v}$ and *z* may combine to make firm A's best response function discontinuous, the root reason for the non-existence of a pure-strategy Nash equilibrium.

Nash equilibrium when *v* and θ are positively correlated

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[26] If firm B sets its price above one, it cannot make any positive sales.

Since in reality consumers with high v tend to have high θ, let us study two cases in which θ and v are positively correlated.

(1) If θ and v are linearly related, say, $v = s\theta$, then, the two-dimensional model is structurally identical to our basic model because $V_A(\theta, v) = \tilde{z}\theta - p_A$ and $V_{A+B}(\theta, v) = \tilde{q}\theta - p_A - p_B$, where $\tilde{z} = z + s$ and $\tilde{q} = q + s$. The results obtained from our basic model in Sections 2-4 apply directly to this case.

(2) Suppose that $v = s\theta + \varepsilon$, where ε is uniformly distributed between $[0, \bar{\varepsilon}]$. Then $V_A(\theta, v) = (z+s)\theta - p_A + \varepsilon$, and $V_{A+B}(\theta, v) = (q+s)\theta - p_A - p_B + \varepsilon$, or equivalently $V_A(\theta) = \varepsilon + \tilde{z}\theta - p_A$, and $V_{A+B}(\theta) = \varepsilon + \tilde{q}\theta - p_A - p_B$, where $\tilde{z} = z + s$ and $\tilde{q} = q + s$. It is equivalent to the two-dimensional case analysed above in this section, so we can apply our earlier results to obtain the following: If the dispersion of consumer types as captured by $\bar{\varepsilon}$ is small, then we might not have a pure-strategy Nash equilibrium; if $\bar{\varepsilon}$ is large, then a pure-strategy Nash equilibrium exists.

6. Conclusions

A dominant firm in one layer of a multi-layered system often seeks to extend the functions of its products to include functions that are traditionally covered by firms in other layers. The definition of product boundaries changes continuously as a consequence of vertical competition. In this paper ,we solve for the equilibrium prices and profits of firms A and B for different boundary values z. We have found that if z is low, then the main product A and its enhancer B are always sold as a bundle; if z is sufficiently high, then the two firms behave as if the two products are independent of each other, and the double mark-up problem disappears; over a certain range of intermediate values of z, no pure-strategy Nash equilibrium exists.

When the simple model is extended from one dimension to two dimensions of consumer heterogeneity, the non-existence of pure-strategy equilibrium survives either if the variation in consumer preferences along the basic function v is small, or if the consumers' preferences along both functions (as captured by v and θ) are sufficiently positively correlated.

[27] Under the parameter values, if p_B is higher than 0.294, firm A's optimal pricing results in independent pricing; if p_B is lower than 0.294, firm A's optimal pricing gives rise to the mixed demand regime.

Appendix

The appendix collects the proofs that were omitted from the text.

The increasing hazard rate condition yields strictly quasi-concave profit functions.

Consider a monopolist supplying a product of quality level z. Without any loss of generality, assume that the marginal cost of the product is zero. Let p denote the price of the product and θ denote the firm's cutoff point i.e., the lowest consumer type buying the product. By definition, $\theta = p/z$. Since there is a one-to-one mapping between p and θ, the firm maximizes $z\theta F(\theta)$ by choosing its optimal cutoff point θ. The first-order condition of profit maximization is given by $zF(\theta)+z\theta f(\theta) = 0$. The second derivative of the profit function is $2zf(\theta)+z\theta f'(\theta)$, which upon substitution of the first-order condition becomes $2zf(\theta)-zF(\theta)f'(\theta)/f(\theta)$.

The increasing hazard rate condition (i.e., $-f/F$ is increasing) implies $(f(\theta))^2-F(\theta)(f'(\theta))>0$, which in turn implies $2zf(\theta)-zF(\theta)f'(\theta)/f(\theta)<0$ because $f(\theta)<0$. That is to say, the profit function is locally strictly concave in θ whenever its slope is zero, i.e., the function is strictly quasi-concave in θ. Q.E.D

Proof of Lemma 1

Firm B maximizes $\Pi_{B1} = F(\frac{p_B}{q-z})(p_B-c_B)$ subject to $p_B \geq \frac{q-z}{z} p_A$ but maximizes $\Pi_{B2} = F(\frac{p_A+p_B}{q})(p_B-c_B)$ subject to $p_B \leq \frac{q-z}{z} p_A$. Under the increasing hazard rate condition, each of these two functions is strictly quasi-concave in p_B and has a single peak. Let p^*_{B1} and $R_B(p_A)$ denote the unconstrained optimal p_B for the profit functions, $\Pi_{B1}(p_B)$ and $\Pi_{B2}(p_B)$, respectively. The first-order conditions of these profit functions are as follows:

$$\frac{\partial \Pi_{B1}}{\partial p_B} = F(\frac{p_B}{q-z}) + f(\frac{p_B}{q-z})(p_B-c_B)\frac{1}{q-z} = 0$$

$$\frac{\partial \Pi_{B2}}{\partial p_B} = F(\frac{p_A+p_B}{q}) + f(\frac{p_A+p_B}{q})(p_B-c_B)\frac{1}{q} = 0$$

The two first-order conditions evaluated at the kink $p_B = \frac{q-z}{z} p_A$ become

$$p_A + \frac{F(\frac{p_A}{z})}{f(\frac{p_A}{z})} z = \frac{z}{q-z} c_B \qquad \text{(A1)}$$

$$p_A + \frac{F(\frac{p_A}{z})}{f(\frac{p_A}{z})} \frac{qz}{q-z} = \frac{z}{q-z} c_B, \tag{A2}$$

Since $F(\theta)/f(\theta)$ is increasing in θ, the left-hand side of (A1) and (A2) is monotonically increasing in p_A. Thus, given q, z, and c_B, there is a unique p_A that solves (A1) or (A2), respectively. Denote the unique p_A satisfying equations (A1) and (A2) by $\underline{p}_A$ and $\overline{p}_A$, respectively.

Since the right-hand side of (A1) is positive and $f(\theta)<0$, we know that $\underline{p}_A > 0$. Also, since $\frac{q-c_B}{z}>1$ and $F(\frac{q-c_B}{z}) = 0$, we have $\overline{p}_A < q$-c_B. Since $f(\theta) <0$ and $z < \frac{qz}{q-z}$, we have $0< \underline{p}_A < \overline{p}_A < q$-$c_B$.

There are three cases:

Case1 ($p_A \leq \underline{p}_A$): When evaluated at $p_B = \frac{q-z}{z} p_A$, we have $\frac{\partial \Pi_{B1}}{\partial p_B} \geq 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} > 0$ because $f(\theta)<0$. Since $\frac{\partial \Pi_{B1}}{\partial p_B} \geq 0$, p^*_{B1} that maximizes Π_{B1} does not violate the constraint that $p_B \geq \frac{q-z}{z} p_A$. However, $R_B(p_A)$ that maximizes Π_{B2} violates the constraint that $p_B \leq \frac{q-z}{z} p_A$. Because $\Pi_{B1}(\frac{q-z}{z} p_A) = \Pi_{B2}(\frac{q-z}{z} p_A)$, the overall profit function has a single peak at p^*_{B1}, and firm B's global optimum price is given by p^*_{B1}.

Case 2 ($\overline{p}_A \leq p_A$): In this case, when evaluated at $p_B = \frac{q-z}{z} p_A$, we have $\frac{\partial \Pi_{B1}}{\partial p_B} <0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} \leq 0$. An argument similar to that for Case 1 establishes that the global optimum price is given by $R_B(p_A)$.

Case 3 ($\underline{p}_A < p_A < \overline{p}_A$): In this case, when evaluated at $p_B = \frac{q-z}{z} p_A$, we have $\frac{\partial \Pi_{B1}}{\partial p_B} <0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} >0$, so the global maximum is achieved at the kink, i.e., $p^*_B = \frac{q-z}{z} p_A$.

When $p_A = \underline{p}_A$, from the definition of $\underline{p}_A$, $\frac{\partial \Pi_{B1}}{\partial p_B} = 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} >0$ at the kind point $p_B = \frac{q-z}{z} p_A$. Thus, the kink is firm B's optimal price. When $p_A = \overline{p}_A$, from the definition of $\overline{p}_A$,

$\frac{\partial \Pi_{B1}}{\partial p_B} < 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B}$ =0 at the kind point $p_B = \frac{q-z}{z} p_A$. Thus, the kink is firm B's optimal price. Therefore, firm B's optimal price is continuous in p_A.

Q. E. D

Proof of Lemma 2:

Firm A's profit function is composed of the two underlying profit functions: $\Pi_{A1} = F(\frac{p_A}{z})(p_A - c_A)$ and $\Pi_{A2} = F(\frac{p_A + p_B}{q})(p_A - c_A)$. The kink of product A's demand curve occurs at $p_A = \frac{z}{q-z} p_B$. Let p^*_{A1} and $R_A(p_B)$ be the unconstrained optimal prices corresponding to Π_{A1} and Π_{A2}, respectively.

Evaluating the two first-order conditions of Π_{A1} and Π_{A2}, respectively, at the kink and rearranging yields

$$p_B + (q-z)\frac{F(\frac{p_B}{q-z})}{f(\frac{p_B}{q-z})} = \frac{q-z}{z} c_A \qquad (A3)$$

$$p_B + q\frac{q-z}{z}\frac{F(\frac{p_B}{q-z})}{f(\frac{p_B}{q-z})} = \frac{q-z}{z} c_A \qquad (A4)$$

Since $F(\theta)/f(\theta)$ is increasing in θ, the left-hand side of (A3) and (A4) is monotonically increasing in p_B. Thus, there exist unique $\underline{p}_B$ and $\overline{p}_B$ that solve (A3) and (A4), respectively. Since f(θ) is negative and q>z, we have $0 < \underline{p}_B < \overline{p}_B$ and there are three cases:

Case1 ($p_B \le \underline{p}_B$): When evaluated at the kink $p_A = \frac{z}{q-z} p_B$, we have $\frac{\partial \Pi_{A1}}{\partial p_A} \ge 0$ and $\frac{\partial \Pi_{A2}}{\partial p_A} > 0$. It implies that p^*_{A1} that maximizes Π_{A1} is equal to or larger than $\frac{z}{q-z} p_B$, while $R_A(p_B)$ that maximizes Π_{A2} does not violates the condition $p_A \ge \frac{z}{q-z} p_B$. Because $\Pi_{A1}(\frac{z}{q-z} p_B) = \Pi_{A2}(\frac{z}{q-z} p_B)$, firm A's global optimum is given by the peak of Π_{A2} at $R_A(p_B)$. Also, since $\frac{\partial \Pi_{A2}}{\partial p_A} > 0$ at the kink, $R_A(p_B)$ is strictly larger than $\frac{z}{q-z} p_B$, and we have $\theta_A > \theta_B$.

Case 2 ($\overline{p}_B \le p_B$) : Again evaluated at the kink, $\frac{\partial \Pi_{A1}}{\partial p_A}$<0 and $\frac{\partial \Pi_{A2}}{\partial p_A} \le 0$. An argument similar to that for case 1 establishes that firm A's globally optimal price is p^*_{A1} . Also, since $\frac{\partial \Pi_{A1}}{\partial p_A}$<0 at the kink, p^*_{A1} is strictly less than $\frac{z}{q-z} p_B$, and we have $\theta_A < \theta_B$.

Case 3 ($\underline{p}_B < p_B < \overline{p}_B$): When evaluated at the kink, we have $\frac{\partial \Pi_{A1}}{\partial p_A}$<0 and $\frac{\partial \Pi_{A2}}{\partial p_A}$ >0, and the overall profit function has two peaks, one at p^*_{A1} and the other at $R_A(p_B)$, where $p^*_{A1} < \frac{z}{q-z} p_B$ $<R_A(p_B)$. We need to compare $\Pi_{A1}(p^*_{A1}; z)$ and $\Pi_{A2}(R_A(p_B); p_B)$ to ascertain firm A's globally optimal price. From case 2, firm A's global optimization occurs at p^*_{A1} when $p_B = \overline{p}_B$, which indicates $\Pi_{A1}(p^*_{A1}; z) > \Pi_{A2}(R_A(p_B); \overline{p}_B)$. From case 1 firm A's global optimization occurs at $R_A(p_B)$ when $p_B = \underline{p}_B$, which indicates $\Pi_{A2}(R_A(p_B); \underline{p}_B) > \Pi_{A1}(p^*_{A1}; z)$. Since $\Pi_{A2}(R_A(p_B); p_B)$ is continuously decreasing in p_B while $\Pi_{A1}(p^*_{A1}; z)$ is independent of p_B, there exists a unique $\tilde{p}_B$ strictly between $\underline{p}_B$ and $\overline{p}_B$ such that $\Pi_{A1}(p^*_{A1}; z) = \Pi_{A2}(R_A(p_B); \tilde{p}_B)$; for $p_B < \tilde{p}_B$, $\Pi_{A1}(p^*_{A1}; z) <$ $\Pi_{A2}(R_A(p_B); p_B)$; for $p_B > \tilde{p}_B$, $\Pi_{A1}(p^*_{A1}; z) > \Pi_{A2}(R_A(p_B); p_B)$.

Combining the above cases yields Lemma 2. Q.E.D.

Proof of Proposition 1

From Corollary 1, we know that there are only two candidates for a pure-strategy Nash equilibrium, (p^*_{A1} , p^*_{B1}) and (p^*_{A2} , p^*_{B2}). Without any loss of generality, let us set q =1 in this proof.

(A). Conditions under which (p^*_{A2} , p^*_{B2}) is a Nash equilibrium:

Suppose that the two firms set their prices at p^*_{A2} and p^*_{B2}, respectively. Then, a necessary condition for these prices to be a Nash equilibrium is that the two firms face demand system (2), or we have to have $\theta_B < \theta_A$, or equivalently $z < \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$. Let us check whether the firms have any incentive to deviate from the prices when $z < \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$.

First, when firm A sets its price at p^*_{A2}, firm B maximizes $\Pi_{B1}(p_B; z)$ if $p_B \ge \frac{1-z}{z} p^*_{A2}$ and $\Pi_{B2}(p_B; p_A)$ if $p_B \le \frac{1-z}{z} p^*_{A2}$. As Lemma 1 shows, firm B's overall profit function has only one peak. Thus, if p^*_{B2} maximizes Π_{B2} without violating the constraint $p_B \le \frac{1-z}{z} p^*_{A2}$, then it is firm B's globally optimal price. When $z < \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$, we have $\frac{p^*_{B2}}{1-z} < \frac{p^*_{A2}}{z}$. Thus, p^*_{B2} maximizes Π_{B2} without violating the constraint and, thus, is firm B's globally optimal price when $p_A = p^*_{A2}$.

Second, given p^*_{B2}, firm A maximizes $\Pi_{A1}= F(\frac{p_A}{z})(p_A-c_A)$ if $p_A \le \frac{z}{1-z} p^*_{B2}$ and $\Pi_{A2}= F(p_A + p^*_{B2})(p_A-c_A)$ if $p_A > \frac{z}{1-z} p^*_{B2}$. Let $S^*_1(z)$ and S^*_2 denote the maximized value of the respective profit functions given the constraints. Since $z < \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$, S^*_2 is equal to $\Pi^*_{A2}= F(p^*_{A2} + p^*_{B2})(p^*_{A2} - c_A)$, which does not depends on z. Let S(z) be defined as $S(z)= S^*_1(z)- S^*_2$, which is continuous and monotonically increasing in z. Let us check the sign of S(z).

If z is smaller than c_A, $S(z) <0$ because $S^*_1(z)$ is zero.

Let us check the sign of $S(\frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}})$. When $z = \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$, we have $p^*_{A2} = \frac{z}{1-z} p^*_{B2}$. Thus, Π_{A2} is optimized at the kink point, $p_A= \frac{z}{1-z} p^*_{B2}$. As Lemma 2 implies, when we have $\frac{\partial \Pi_{A2}}{\partial p_A} = 0$ at the kink point, we have $\frac{\partial \Pi_{A1}}{\partial p_A} <0$ at the kink point. Thus, $S(\frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}) >0$.

Since S(z) is continuous and monotonically increasing in z, there exists a unique z_{min} such that $S(z_{min}) =0$. Also, since $S(\frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}})>0$, we have $z_{min} < \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$, which implies that $\theta_B < \theta_A$ for $z \le z_{min}$. In summary, for $z \le z_{min}$ the two firms do not deviate from (p^*_{A2} and p^*_{B2}), and these prices constitute a Nash equilibrium.

(B). Conditions under which (p^*_{A1}, p^*_{B1}) is a Nash equilibrium:

Suppose the two firms set their prices at p^*_{A1} and p^*_{B1}, respectively. Then, a necessary condition for these prices to be a Nash equilibrium is that the two firms face demand system (1). That is, under the prices we have to have $\theta_B > \theta_A$, or equivalently $\frac{p^*_{B1}}{1-z} > \frac{p^*_{A1}}{z}$. Since $\frac{p^*_{A1}}{z}$ (*resp.* $\frac{p^*_{B1}}{1-z}$) depends only on $\frac{c_A}{z}$ (*resp.* $\frac{c_B}{1-z}$), we have $\frac{p^*_{B1}}{1-z} > \frac{p^*_{A1}}{z}$ if and only if $z > \frac{c_A}{c_A+c_B}$. Let us check whether the firms have any incentive to deviate from (p^*_{A1}, p^*_{B1}) when $z > \frac{c_A}{c_A+c_B}$ or equivalently when $\frac{p^*_{B1}}{1-z} > \frac{p^*_{A1}}{z}$.

First, firm B maximizes $\Pi_{B1}(p_B; z)$ if $p_B \geq \frac{1-z}{z} p^*_{A1}$ and $\Pi_{B2}(p_B; p_A)$ if $p_B \leq \frac{1-z}{z} p^*_{A1}$. As Lemma 1 shows, firm B's overall profit function has only one peak. When $z > \frac{c_A}{c_A+c_B}$, we have $\frac{p^*_{B1}}{1-z} > \frac{p^*_{A1}}{z}$, and p^*_{B1} maximizes Π_{B1} without violating the constraint. Thus, firm B's global peak occurs at p^*_{B1}, and firm B would not deviate from p^*_{B1} if $p_A = p^*_{A1}$.

Second, let us check whether p^*_{A1} is firm A's optimal price, given p^*_{B1}. Firm A maximizes $\Pi_{A1} = F(\frac{p_A}{z})(p_A\text{-}c_A)$ if $p_A \leq \frac{z}{1-z} p^*_{B1}$ and $\Pi_{A2} = F(p_A + p^*_{B1})(p_A\text{-}c_A)$ if $p_A \geq \frac{z}{1-z} p^*_{B1}$. Let $T^*_1(z)$ and $T^*_2(z)$ denote the maximized values of the respective profit functions given the constraints. Similar to part (A), let $T(z)$ be defined as $T(z) = T^*_1(z) - T^*_2(z)$. Since $\frac{p^*_{B1}}{1-z} > \frac{p^*_{A1}}{z}$, $T^*_1(z)$ is equal to Π^*_{A1}, which is increasing in z. Since p^*_{B1} is decreasing in z, $T^*_2(z)$ is a function of z, too. Since both functions are continuous in z, so is $T(z)$.

If $z = \frac{c_A}{c_A+c_B}$, then we have $\frac{p^*_{A1}}{z} = \frac{p^*_{B1}}{1-z}$, and Π_{A1} is optimized at the kink $\frac{z}{1-z} p^*_{B1}$. According to Lemma 2, when $\frac{\partial \Pi_{A1}}{\partial p_A} = 0$ at the kink point, we have $\frac{\partial \Pi_{A2}}{\partial p_A} > 0$ at the kink point. Thus, given p^*_{B1}, firm A's optimal price is $R_A(p^*_{B1})$, which implies that when $z = \frac{c_A}{c_A+c_B}$ we have $T(z) < 0$ and that firm A wants to deviate from p^*_{A1}.

To establish the sign of $T(z)$ at $z = 1\text{-}c_B$, we check the first-order condition of Π_{A2} when evaluated at the kink $p_A = \frac{z}{1-z} p^*_{B1}$. The first-order condition at the kink is $\frac{\partial \Pi_{A2}}{\partial p_A} = F(\theta^*_{B1}) + f(\theta^*_{B1})(z\theta^*_{B1} - c_A)$, which when combined with the first-order condition of Π_{B1} evaluated at p^*_{B1}, becomes $-f(\theta^*_{B1})((1-z)\theta^*_{B1} + c_A - \frac{c_B}{1-z})$. Since $f(\theta) < 0$, the sign of the expression is equal to that of $((1-z)\theta^*_{B1} + c_A - \frac{c_B}{1-z})$, which is monotonically decreasing in z. When $z = (1\text{-}c_B)$, the sign of the expression is equal to that of $[c_B\theta_B + c_A - 1]$, which is negative because $c_B + c_A < 1$ and $\theta_B < 1$. As Lemma 2 shows, when $\frac{\partial \Pi_{A2}}{\partial p_A} < 0$ at the kink, firm A's global optimal price is given by p^*_{A1}. Thus, when $z = 1\text{-}c_B$, we have $T(z) > 0$ and firm A does not want to deviate from p^*_{A1}.

Since $T(\frac{c_A}{c_A+c_B}) < 0$, $T(1\text{-}c_B) > 0$ and $T(z)$ is continuous in z, there exists at least one solution to $T(z) = 0$. Since $T(\frac{c_A}{c_A+c_B}) < 0$, the solutions to $T(z) = 0$ are larger than $\frac{c_A}{c_A+c_B}$. If $T(z)$ has multiple solutions, let z_{max} denote the largest solution. By definition $T(z) > 0$ for all $z > z_{max}$, implying that firm A would not switch from independent pricing to bundling pricing if $z > z_{max}$.

In summary, since $z_{max} > \frac{c_A}{c_A+c_B}$, if $z \geq z_{max}$, then we have $\theta_B > \theta_A$, and the two firms would not deviate from (p^*_{A1}, p^*_{B1}). That is, these prices constitute a Nash equilibrium.

(C). $z_{min} < z_{max}$

From the first-order conditions of p^*_{A2} and p^*_{B2}, we have

$$\frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}} = \frac{c_A - \frac{F(p^*_{A2}+p^*_{B2})}{f(p^*_{A2}+p^*_{B2})}}{c_A + c_B - 2\frac{F(p^*_{A2}+p^*_{B2})}{f(p^*_{A2}+p^*_{B2})}}$$

. Since $f(\theta)<0$, we obtain $\frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}} < \frac{c_A}{c_A+c_B}$. But we have already shown that $z_{min} < \frac{p^*_{A2}}{p^*_{A2}+p^*_{B2}}$ and that all solutions to $T(z)=0$ are larger than $\frac{c_A}{c_A+c_B}$. Combining the inequalities yields $z_{min} < \frac{c_A}{c_A+c_B} < z_{max.}$

If the solution of $T(z) = 0$ is unique, then it is equal to z_{max}, and for all z between z_{min} and z_{max}, $T(z)<0$, which indicates that there is no pure-strategy Nash equilibrium for those values of z.

However, if there are multiple solutions, since $T(z) <0$ for small z and $T(z) >0$ for large z, $T(z) = 0$ has an odd number of solutions, and there are alternating sub-intervals over which $T(z) > 0$ and sub-intervals over which $T(z)<0$. If $z_{min}< z$ and $T(z) >0$, then independent pricing is Nash equilibrium for the values of z. If $z_{min}< z$ and $T(z) <0$, then there is no pure-strategy Nash equilibrium for the values of z. Nevertheless, a mixed-strategy Nash equilibrium can be constructed.

(D). Mixed-strategy Nash Equilibrium:

The firms' best response curves as depicted below help to demonstrate the construction of the mixed-strategy Nash equilibrium. Suppose that firm B sets its price at $\tilde{p}_B$ such that firm A has two optimal prices and is indifferent between p^*_{A1} and $R_A(\tilde{p}_B)$. Suppose that firm A sets its price at p^*_{A1} with probability α and at $R_A(\tilde{p}_B)$ with probability $(1-\alpha)$. Let us analyse whether firm B's best response to this mixed strategy would indeed be $\tilde{p}_B$.

The existence and characteristics of the Nash equilibrium depend on where the two points, *m* and *n*, in Figure A(1) are located in Figure A(2). Let y_1 and y_2 denote firm B's best response to

p^*_{A1} and $R_A(\tilde{p}_B)$, respectively. Since $\frac{1}{|R'_A|} > |R'_B|$, when there is no pure-strategy Nash equilibrium, point *n* in Figure 7-(a) must be in area B in Figure 7-(b). It indicates that $\tilde{p}_B < y_2$.

If y_1 is higher than or equal to $\tilde{p}_B$, then firm A's best response to y_1 is p^*_{A1}, and (p^*_{A1}, y_1) becomes a pure-strategy equilibrium, which contradicts the non-existence of such an equilibrium. Thus, point *m* must be in area C, which indicates that we have $y_1 < \tilde{p}_B$.

Regardless of whether firm A sets its price p_{A1} or $R_A(\tilde{p}_B)$, we can show that in terms of firm B's profit any p_B higher than y_2 is dominated by y_2 and that any p_B lower than y_1 is dominated by y_1. Thus, firm B's optimal price to firm A's randomization is strictly in between y_1 and y_2.

When point *n* is in area B and point *m* is in area C, if firm B sets its price p_B strictly in between y_1 and y_2, we have $\frac{q-z}{z} p_{A1} < p_B < \frac{q-z}{z} R_A(\tilde{p}_B)$, implying that firm B's profit function is given by

$$\Pi_B(p_B) = \alpha F(\frac{p_B}{q-z})(p_B - c_B) + (1-\alpha)F(\frac{R_A(\tilde{p}_B)+p_B}{q})(p_B - c_B)$$

If $\alpha=0$, firm B's optimal price is y_2; if $\alpha=1$, firm B's optimal price is y_1. Since $\tilde{p}_B$ is between y_1 and y_2 and firm B's optimal price is continuous in α, by the mean value theorem there exists α such that firm B's optimal price is equal to $\tilde{p}_B$. This proves that [α, p_{A1}, p_{A2}, $\tilde{p}_B$] constitute a mixed-strategy Nash Equilibrium. Q.E.D.

Proof of Proposition 3

Part (a):

Firm A's optimal price satisfies the following first-order condition:

$$p^*_A = c_A - z\frac{F(\theta_A)}{f(\theta_A)}$$

Since $(-\frac{F(\theta_A)}{f(\theta_A)})$ is decreasing in θ_A and θ_A is decreasing in *z*, the right-hand side of the above condition is increasing in *z*, i.e., $p^*_{A1}(z)$ is increasing in *z* for all $z \geq z_{max}$.

Firm B's optimal price satisfies the following first-order condition:

$$p^*_B = c_B - (q-z)\frac{F(\theta_B)}{f(\theta_B)}$$

Since $(-\frac{F(\theta_B)}{f(\theta_B)})$ is decreasing in θ_B and θ_B is increasing in *z*, the right-hand side of the above condition is decreasing in *z*. Thus, $p^*_{B1}(z)$ decreases in *z* for all $z \geq z_{max}$

Part (b) follows from changes in θ^*_A and θ^*_B.

Part (c) :

As firm A maximizes $zF(\theta_A)(\theta_A - \frac{c_A}{z})$, its profit clearly increases in *z* for $z > z_{max}$. As firm B maximizes $(q-z)F(\theta_B)(\theta_B - \frac{c_B}{q-z})$, its profit deceases in *z* for $z > z_{max}$.

If $z > z_{max}$, the two parts of the industry profits are independent of each other, and by the envelope theorem, we obtain $\frac{\partial(\Pi_A+\Pi_B)}{\partial z} = F(x_A^*)(x_A^*) - F(x_B^*)(x_B^*)$. Please note that x_A^* maximizes $F(x_A)(x_A - \frac{c_A}{z})$, and x_B^* maximizes $F(x_B)(x_B - \frac{c_B}{q-z})$. (Note that $F(x)(x)$ is revenue for a firm maximizing $F(x)(x-c)$). As a firm's marginal cost gets lower, its revenue gets higher. That is, if $\frac{c_A}{z} < \frac{c_B}{q-z}$, or equivalently if $z > \frac{qc_A}{c_A+c_B}$, then $F(x_A^*)(x_A^*) > F(x_B^*)(x_B^*)$. Thus, $\frac{\partial(\Pi_A+\Pi_B)}{\partial z} > 0$. Q. E. D

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Figures

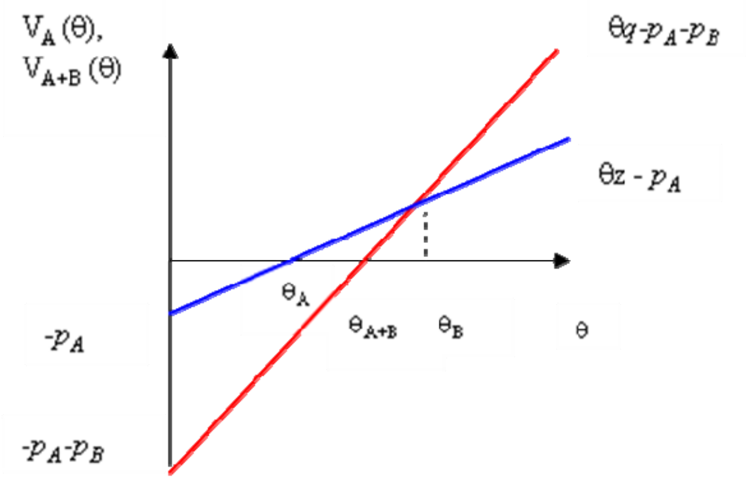

Figure 1. $\theta_A < \theta_{A+B} < \theta_B$ Virtually Independent Products

Jae Nahm, Figure 1 of 7.

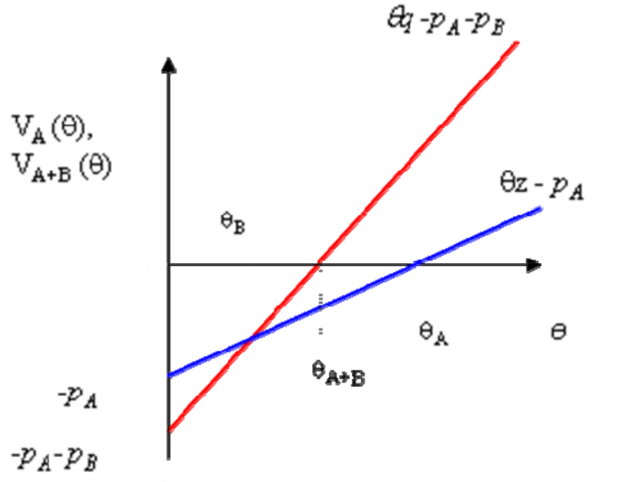

Figure 2. $\theta_B < \theta_{A+B} < \theta_A$, Virtually Strict Complements

Jae Nahm, Figure 2 of 7.

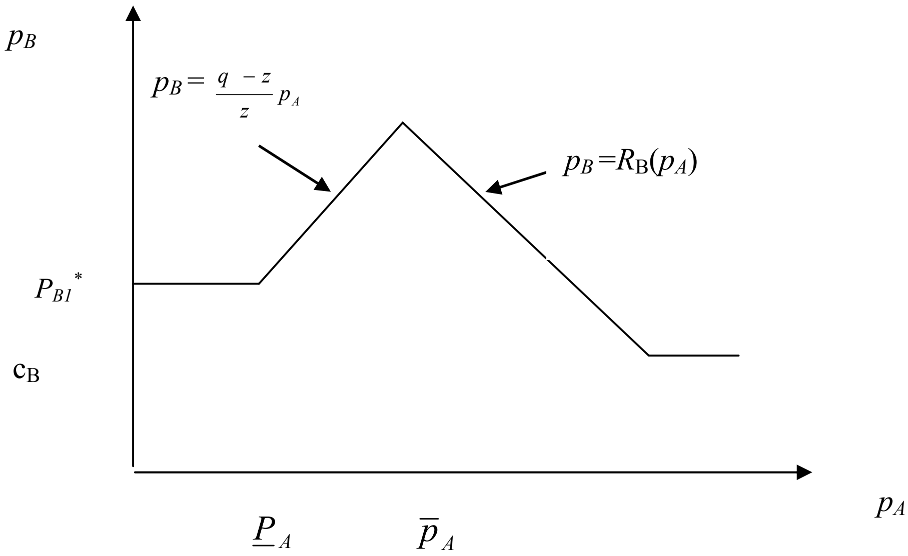

Figure 3. Firm B's best response to p_A if θ is uniformly distributed

Jae Nahm, Figure 3 of 7.

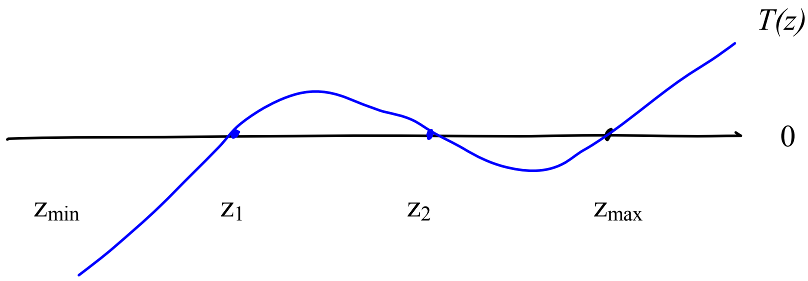

Figure 4. The sign of *T(z)* when *T(z)*=0 has multiple solutions

Jae Nahm, Figure 4 of 7.

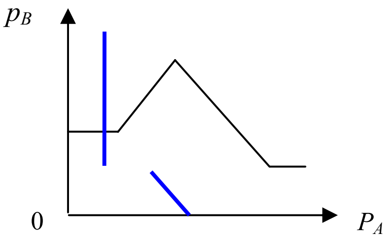

Figure 5-(a), z ≥z_{max}

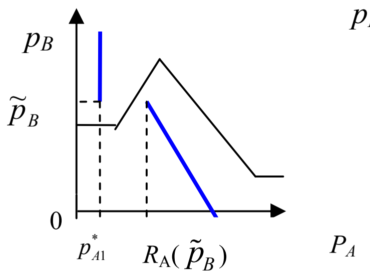

Figure 5-(b) $z_{min} < z < z_{max}$

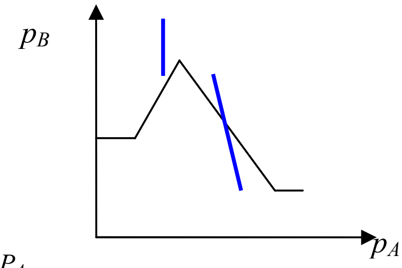

Figure 5-(c), z ≤ z_{min}

Figure 5 (a), (b), and (c), Nash equilibrium when θ is uniformly distributed

Jae Nahm, Figure 5 of 7.

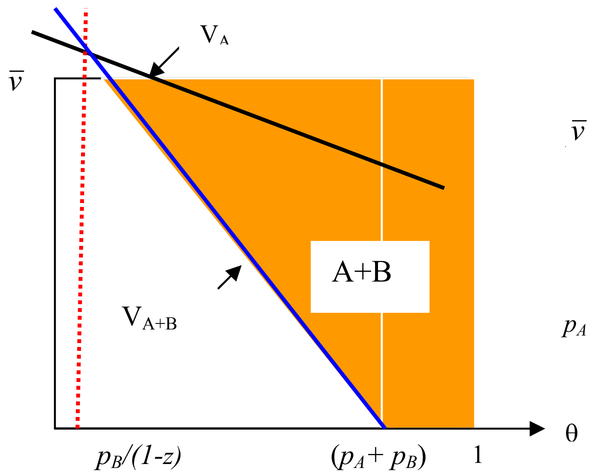

Figure 6- (a)

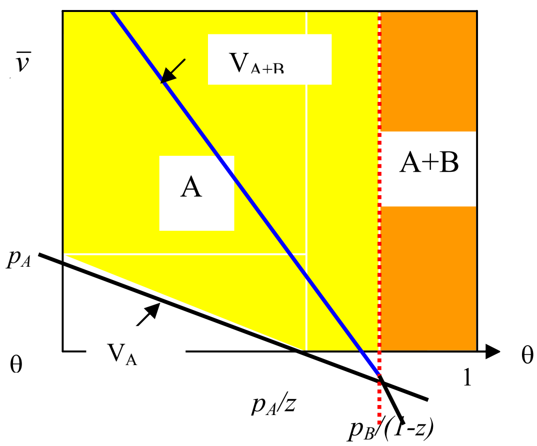

Figure 6- (b)

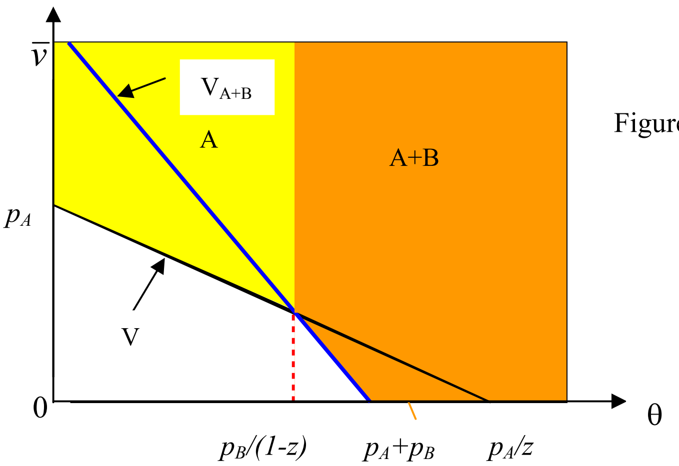

Figure 6- (c).

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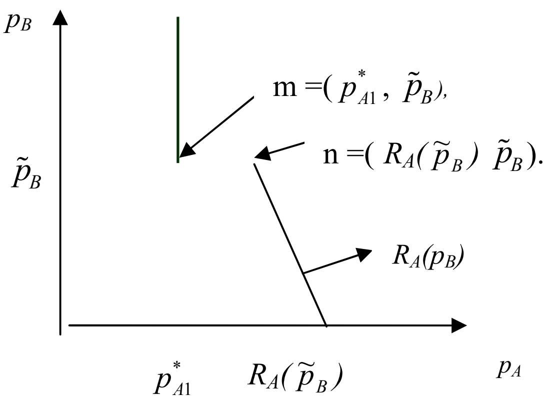

Figure A (1). Firm A's best response curve

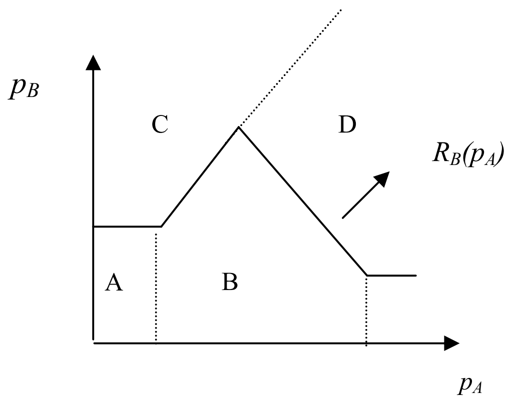

Figure A (2). Firm B's best response

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