Higher-Order Risk Attitudes toward Correlation

Jingyuan Li

Department of Finance and Insurance, Lingnan University

July 29, 2013

Covariance

- $\tilde{x}$ and $\tilde{y}$ are valued in the intervals $[a, b]$ and $[c, d]$.
-
$$Cov(\tilde{x}, \tilde{y}) = E\tilde{x}\tilde{y} - E\tilde{x}E\tilde{y}. \tag{1}$$
- $E\tilde{x}\tilde{y} \geq E\tilde{x}E\tilde{y}$ if and only if $\tilde{x}$ and $\tilde{y}$ co-vary positively.

Equity premium

- u: the bivariate utility of the representative agent
- $\tilde{x}$: the GDP per capita
- $\tilde{y}$: the background risk
- Equity premium, φ:

$$\varphi = \frac{E\tilde{x}Eu^{(1,0)}(\tilde{x},\tilde{y})}{E\tilde{x}u^{(1,0)}(\tilde{x},\tilde{y})} - 1. \tag{2}$$

- $u^{(k_1,k_2)}$: the (k_1, k_2)th cross derivative of u, i.e., $u^{(k_1,k_2)} = \frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial y^{k_2}}u(x,y)$

Signing equity premium

- Full information: e.g., $u(x, y) = \log(x + y)$ and $(\tilde{x}, \tilde{y})$ is joint-normal distributed $\Rightarrow$ $\mathit{sign}(\varphi)$
- Partial information: e.g., risk aversion and $(\tilde{x}, \tilde{y})$ is affiliated $\Rightarrow$ $\mathit{sign}(\varphi)$?
- Sign the covariance between functions

Covariance between monotonic functions

- **Definition** (Esary et al. 1967, p1466) $(\tilde{x}, \tilde{y})$ is said to be associated if for all functions α, β which are increasing in each component,

$$Cov(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y})) \geq 0. \tag{3}$$

- Risk averse in x ($u^{(2,0)} < 0$), correlation averse ($u^{(1,1)} \leq 0$) and $(\tilde{x}, \tilde{y})$ is associated $\Rightarrow \varphi \geq 0$

Higher-order risk attitudes

- Higher-order risk attitudes (e. g., prudence and temperance)$\Leftrightarrow$signing the higher-order derivative of the utility function (expected utility framework)
- Weaker dependence structures of $(\tilde{x}, \tilde{y})$

N^{th}-order stochastic dominance dependence

- **Definition** Define $F^1(y|x) = F(y|x)$ and $F^{n+1}(y|x) = \int_c^y F^n(t|x)dt$. We say that $\tilde{y}$ is N^{th}-order stochastic dominance dependent on x ($N^{th}SDD(\tilde{y}|x)$) if
(i) $F^N(y|x') \leq F^N(y|x)$ for all y and $x' \geq x$;
(ii) $F^n(d|x') \leq F^n(d|x)$ for all y, $x' \geq x$ and $n = 1, ...N - 1$.

- Eeckhoudt and Kimball (1992): Third-order stochastic dominance dependence

Covariance between functions beyond monotonicity

- **Definition** (Denuit et al. 1999) The class $\mathcal{U}_{(s_1,s_2)-icv}$ of the regular (s_1, s_2)-increasing concave functions defined as the class of all the functions u, for s_1 and s_2 are positive integers, such that $(-1)^{k_1+k_2+1}u^{(k_1,k_2)} \geq 0$ for all $k_1 = 0, 1, ..., s_1$, $k_2 = 0, 1, ..., s_2$ with $k_1 + k_2 \geq 1$.

- **Proposition**
  The following statements are equivalent.
  (i)
  $$Cov(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y})) \geq 0 \tag{4}$$
  for all α and β such that $\alpha^{(1,0)} \geq 0$, $\beta^{(1,0)} \geq 0$, $\alpha \in \mathcal{U}_{(0,I)-icv}$ and $\beta \in \mathcal{U}_{(0,J)-icv}$;
  (ii) $N^{th}SDD(\tilde{y}|x)$ where $N = \min(I, J)$.

Various concepts of bivariate dependence

-

$$\begin{aligned} & (\tilde{x}, \tilde{y}) \textit{ is associated} \qquad (5) \\ \Rightarrow \quad & N^{th} SDD(\tilde{y}|x) \\ \Rightarrow \quad & \textit{positive } N^{th} ED(\tilde{y}|x) \end{aligned}$$

-

$$\textit{positive SED on } \tilde{y} \Rightarrow (\tilde{x}, \tilde{y}) \textit{ is positive correlated} \qquad (6)$$

Risk aversion in the presence of another risk

- **Proposition**(Finkelshtain et al. 1999, part (a) and (b) of Theorem 2)
  The following statements are equivalent.
  (i) $Eu(\tilde{x}, \tilde{y}) \leq Eu(E\tilde{x}, \tilde{y})$ for $(\tilde{x}, \tilde{y})$ such that $E(\tilde{x}|\tilde{y})$ is increasing in y;
  (ii) $u^{(2,0)} \leq 0$ and $u^{(1,1)} \leq 0$.

Risk aversion in the presence of another risk

- **Proposition**
  Suppose $N^{th}SDD(\tilde{y}|x)$, $u^{(2,0)} \leq 0$ and $-u^{(1,0)} \in \mathcal{U}_{(0,N)-icv}$, then
  $$Eu(\tilde{x},\tilde{y}) \leq Eu(E\tilde{x},\tilde{y}). \tag{7}$$

If one of the following conditions is satisfied, then an agent is risk averse for $\tilde{x}$ in the presence of $\tilde{y}$:
(i) she is risk averse in x ($u^{(2,0)} \leq 0$), correlation averse ($u^{(1,1)} \leq 0$) and $FSDD(\tilde{y}|x)$;
(ii) she is risk averse in x ($u^{(2,0)} \leq 0$), correlation averse $u^{(1,1)} \leq 0$, cross-prudent in x ($u^{(1,2)} \geq 0$) and $SSDD(\tilde{y}|x)$;
(iii) she is risk averse in x ($u^{(2,0)} \leq 0$), correlation averse $u^{(1,1)} \leq 0$, cross-prudent and cross-temperate in x ($u^{(1,2)} \geq 0$ and $u^{(1,3)} \leq 0$), and $TSDD(\tilde{y}|x)$.

Application

- $$U(s) = u_0(x_0 - s, h_0) + \frac{1}{1+\rho} E u_1(s(1+r) + \tilde{x}, \tilde{h}) \qquad (8)$$

$$u_0^{(1,0)}(x_0 - s^*, h_0) = \frac{1+r}{1+\rho} E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, \tilde{h}) \qquad (9)$$

$s_{x,E}$: the solution with $(\tilde{x}, E\tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$
$s_{E,h}$: the solution with $(E\tilde{x}, \tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$.

- **Proposition**
(i) $u_0^{(2,0)} \leq 0,\ u_1^{(2,0)} \leq 0,\ u_1^{(1,2)} \geq 0,\ u_1^{(1,1)} \in \mathcal{U}_{(N,0)-icv}$ and $N^{th}SDD(\tilde{x}|h) \Rightarrow s^* \geq s_{x,E}$;
(ii) $u_0^{(2,0)} \leq 0,\ u_1^{(3,0)} \geq 0,\ u_1^{(2,0)} \in \mathcal{U}_{(0,N)-icv}$ and $N^{th}SDD(\tilde{h}|x) \Rightarrow s^* \geq s_{E,h}$.

Application

- $$V(a) = u_0(x_0 - a, h_0) + \frac{1}{1+\rho} E u_1(\tilde{x}, \tilde{h} + a) \qquad (10)$$

  The optimal amount of investment a^* is determined by

  $$u_0^{(1,0)}(x_0 - a^*, h_0) = \frac{m}{1+\rho} E u_1^{(0,1)}(\tilde{x}, \tilde{h} + m a^*) \qquad (11)$$

  $a_{x,E}$: the solution with $(\tilde{x}, E\tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$
  $a_{E,h}$: the solution with $(E\tilde{x}, \tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$.

- **Proposition**
  (i) $u_0^{(2,0)} \leq 0$, $u_1^{(0,3)} \geq 0$, $u_1^{(0,2)} \in \mathcal{U}_{(N,0)-icv}$ and $N^{th}SDD(\tilde{x}|h)$ $\Rightarrow a^* \geq a_{x,E}$;
  (ii) $u_0^{(2,0)} \leq 0$, $u_1^{(0,2)} \leq 0$, $u_1^{(2,1)} \geq 0$, $u_1^{(1,1)} \in \mathcal{U}_{(0,N)-icv}$ and $N^{th}SDD(\tilde{h}|x) \Rightarrow a^* \geq a_{E,h}$.

A class of bivariate stochastic orderings

- $(\tilde{x}_1, \tilde{y}_1)$ and $(\tilde{x}_2, \tilde{y}_2)$ are two 2-dimensional random vectors with density functions f and g

$$
\begin{aligned}
Eu(\tilde{x}_1, \tilde{y}_1) &= \int_a^b \int_c^d u(x, y) f(x, y) dxdy \qquad (12) \\
&= \int_a^b \int_c^d u(x, y) \frac{f(x, y)}{g(x, y)} g(x, y) dxdy \\
&= E[u(\tilde{x}_2, \tilde{y}_2) \frac{f(\tilde{x}_2, \tilde{y}_2)}{g(\tilde{x}_2, \tilde{y}_2)}] \\
&= E[u(\tilde{x}_2, \tilde{y}_2)] + Cov[u(\tilde{x}_2, \tilde{y}_2), \frac{f(\tilde{x}_2, \tilde{y}_2)}{g(\tilde{x}_2, \tilde{y}_2)}],
\end{aligned}
$$

If $(\tilde{x}_2, \tilde{y}_2)$ is associated and $\frac{f}{g}$ is increasing in x and y, then $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ (see e.g., Shaked and Shanthikumar 2007, Theorem 6.B.8).

A class of bivariate stochastic orderings

- **Proposition**
  The following statements are equivalent.
  (i) $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ for all u, f and g such that $u^{(1,0)} \geq 0$, $(\frac{f}{g})^{(1,0)} \geq 0$, $u \in \mathcal{U}_{(0,I)-icv}$, $(\frac{f}{g}) \in \mathcal{U}_{(0,J)-icv}$;
  (ii) $N^{th} SDD(\tilde{y}|x)$ where $N = \min(I, J)$.

When $I = J = 2$, $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ if u is monotonic ($u^{(1,0)} \geq 0$ and $u^{(0,1)} \geq 0$) and risk averse in y ($u^{(0,2)} \leq 0$), $\frac{f}{g}$ is increasing in x and y, concave in y, and $SSDD(\tilde{y}|x)$.

Application

- $$U(s) = u_0(x_0 - s, h_0) + \frac{1}{1+\rho} E u_1(s(1+r) + \tilde{x}, \tilde{h}) \tag{13}$$

  $$u_0^{(1,0)}(x_0 - s^*, h_0) = \frac{1+r}{1+\rho} E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, \tilde{h}) \tag{14}$$

  s': the solution with $(\tilde{x}', \tilde{h}')$ substituted for $(\tilde{x}, \tilde{h})$

- **Proposition** $u_0^{(2,0)} \leq 0$, $u_1^{(2,0)} \leq 0$, $(\frac{f}{g})^{(1,0)} \geq 0$, $-u_1^{(1,0)} \in \mathcal{U}_{(0,I)-icv}$, $(\frac{f}{g}) \in \mathcal{U}_{(0,J)-icv}$ and $N^{th} SDD(\tilde{h}'|x')$ where $N = \min(I, J) \Rightarrow s^* \leq s'$;

Application

- $$V(a) = u_0(x_0 - a, h_0) + \frac{1}{1+\rho} E u_1(\tilde{x}, \tilde{h} + a) \tag{15}$$

  The optimal amount of investment a^* is determined by

  $$u_0^{(1,0)}(x_0 - a^*, h_0) = \frac{m}{1+\rho} E u_1^{(0,1)}(\tilde{x}, \tilde{h} + m a^*) \tag{16}$$

  a: the solution of (16) with $(\tilde{x}', \tilde{h}')$ substituted for $(\tilde{x}, \tilde{h})$

- **Proposition** $u_0^{(2,0)} \leq 0$, $u_1^{(0,2)} \leq 0$, $u_1^{(1,1)} \leq 0$, $(\frac{f}{g})^{(1,0)} \geq 0$, $-u_1^{(0,1)} \in \mathcal{U}_{(0,I)-icv}$, $(\frac{f}{g}) \in \mathcal{U}_{(0,J)-icv}$ and $N^{th} SDD(\tilde{h}'|x') \Rightarrow a^* \leq a'$.

Justify the first-order approach to bi-signal principal-agent problems

- $$U(a) = \int_a^b \int_c^d u(s(x, y)) f(x, y|a) dxdy - a \qquad (17)$$
- $U(a)$ is concave $\Rightarrow$ First-order-approach (FOA)
- Monotone likelihood ratio condition (MLRC) and the concavity of the distribution function condition (CDFC) (Rogerson, 1985) $\Rightarrow$ $U(a)$ is concave
- Most of the distribution functions do not have the CDFC property

Justify the first-order approach to bi-signal principal-agent problems

- Jewitt: Define $H(x, y) = u(s(x, y))$.

$$\frac{d^2}{da^2}U(a) = -Cov(H(\tilde{x}, \tilde{y})), -\frac{f_{aa}(\tilde{x}, \tilde{y}|a)}{f(\tilde{x}, \tilde{y}|a)}), \qquad (18)$$

- $(\tilde{x}, \tilde{y})$ is affiliated, $H(x, y)$ and $-\frac{f_{aa}(x,y|a)}{f(x,y|a)}$ are increasing functions $\Rightarrow \frac{d^2}{da^2}U(a) \leq 0$

Justify the first-order approach to bi-signal principal-agent problems

- **Proposition**:
  The following statements are equivalent.
  (i)
  $$\frac{d^2}{da^2}U(a) \leq 0 \tag{19}$$
  for all $H(x,y)$ and $f(x,y|a)$ such that $H^{(1,0)} \geq 0$
  $(\frac{f_{aa}}{f})^{(1,0)} \leq 0$, $H \in \mathcal{U}_{(0,I)-icv}$ and $-\frac{f_{aa}}{f} \in \mathcal{U}_{(0,J)-icv}$;
  (ii) $N^{th}SDD(\tilde{y}|x)$ where $N = \min(I, J)$.