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The Choice of the Number of Varieties: Justifying Simple Mechanisms

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Abstract

We study a mechanism designer's trade-off between the complexity level and optimality level of a mechanism. While our techniques apply to a much larger class of mechanism design problems, we restrict our presentation to Mussa and Rosen (1978) quality differentiation in which a monopolist restricts itself to offering a menu with at most a finite number n of varieties. We prove that (i) the marginal benefit of adding one more variety is diminishing in n; (ii) the loss due to the restriction on the number of varieties is of order no more than $1/n^2$; (iii) the marginal benefit of adding one more variety is of order no more than $1/n^3$; and (iv) offering only two varieties can make more than two-third of the potential profit that can be made by the second best offering. Roughly speaking, our analysis predicts that the monopolist should very plausibly offer only a small number of varieties in the menu.

Keywords: Quality differentiation, Monopoly pricing, Simple mechanisms

JEL Classification Numbers: D42, D82, L15.

1 Introduction

Mechanism design theory has now become a classic and far-reaching branch in economics. It has been used to derive, for example, optimal income taxation scheme (Mirrlees (1971)), optimal nonlinear pricing scheme (Maskin and Riley (1984)), optimal quality differentiation

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(Mussa and Rosen (1978)), among many others. While these theoretical solutions of optimal mechanisms have been well known, in reality however people embrace much simpler mechanisms, like, an income taxation scheme with several tax bands and several marginal tax rates, a multipart tariff with a small number of "parts", and a quality-price scheme with only a few quality-differentiated varieties. How well can a suboptimal but simpler mechanism perform relative to the fully optimal mechanism? If complicating the mechanism is costly, how should the mechanism designer choose the optimal "complexity level" of the mechanism?

We will consider the framework of Mussa and Rosen (1978) monopolistic quality differentiation. In this framework, a monopolist is uninformed about its customers' preferences over quality (or types), but it can produce and offer a spectrum of quality-differentiated varieties to separate different types of its customers. The optimal spectrum involves a continuum of quality-differentiated varieties, tailor-made for each consumer type. However, if the monopolist desires, for practical concerns, to offer at most a finite number n of varieties only, it would design a discrete offering (i.e. a menu of a finite number of quality-price choices), in order to maximize profit subject to the number of varieties n. There would then be a "constrained profit" Π_n for each n. Our main task is to characterize the properties of the constrained profit sequence $\{\Pi_n\}_{n=0}^{\infty}$. We also consider the setting with a fixed cost of developing each variety, which endogenizes the number of varieties. Roughly speaking, our analysis predicts that the monopolist should very plausibly offer only a small number of varieties in the menu.[1]

Although we restrict attention to the monopolistic quality differentiation problem to make our presentation concrete, we emphasize that the techniques developed in this paper can be applied to other mechanism design (or principal-agent) problems, where there is one principal and one agent, and the agent has one-dimensional private information.[2] The number n should be thought of as a measure of complexity level of a mechanism, which could be interpreted in different ways in different kinds of problems. For example, n could be reinterpreted as the number of two-part tariffs offered by the seller to consumers in the context of nonlinear pricing, or the number of possible messages that can be sent from the agent to the principal in the principal-agent models with limited communication.

The "constrained program", i.e. seeking the optimal discrete offering subject to the number of varieties, has no explicit solution unless for special cases (in Subsection 3.2). However, we are able to uncover a number of qualitative features of an optimal discrete offering and the constrained profit sequence $\{\Pi_n\}_{n=0}^{\infty}$. Firstly, it is not hard to show that

[1] This is also true for a social planner's welfare maximization problem.

[2] For the optimal solution of this kind of problems, see Fudenberg and Tirole (1991) Chapter 7, or Guesnerie and Laffont (1984).

an optimal discrete offering (given any n) must be a step function fluctuated around the optimal continuous offering (or second best offering); and Π_n monotonically converges to the fully optimal profit (or second best profit) Π_∞ as n gets large.

If adding every extra variety in the offered menu is costly, and if the marginal benefit of adding one more variety $\Pi_{n+1} - \Pi_n$ is diminishing, then the monopolist should optimally choose the number n of varieties that approximately equalizes the marginal benefit and marginal cost of adding one more variety. Our first main result is that the marginal benefit $\Pi_{n+1} - \Pi_n$ is really diminishing in n. To the best of my knowledge, this is the first diminishing marginal benefit result in any similar context. Intuitively, as the number of varieties that have already been offered is larger, the space for improving profit by adding one more variety is less, and hence the effectiveness of the extra variety is less. However, this "diminishing marginal benefit" property is far from trivial, because adding one more variety would bring about an optimal adjustment of all previously offered varieties. Although $\Pi_n - \Pi_{n-1}$ must ultimately diminish, it is rather surprising that the property holds for every n in general setup. This diminishing marginal benefit property is not only interesting on its own, but also crucial to proving many others of our results. Moreover, the proof of diminishing marginal benefit property, which involves comparing different constrained profits and suboptimal profits in graphs, is very original and also interesting on its own.

Our second main result is what we call "quadratic rate result", i.e. the "uncaptured profit" $\Pi_\infty - \Pi_n$ is of order no more than $1/n^2$. The intuition is that the slope of virtual surplus with respect to quality is flat at the ideal second best quality. Hence, the loss from deviating from the second best quality, due to discrete offering, is of second or higher order, but not of first order. Restricting to a finite number n of varieties, although different types of consumers have to be pooled and served with a single quality, the distance between the quality serving a particular type and the second best quality for that type is approximately proportional to $1/n$. A Taylor expansion argument shows that the uncaptured profit is of order no more than $1/n^2$. Moreover, this convergence rate can be attained by a simple offering rule, which only involves uniformly distributed set of (suboptimal) varieties. Furthermore, the bound we provide for $\Pi_\infty - \Pi_n$ is tight.

Our third main result is what we call "cubic rate result", i.e. the marginal benefit of adding one more variety $\Pi_{n+1} - \Pi_n$ is of order no more than $1/n^3$. As a matter of mathematical fact, the aforementioned quadratic rate result alone does not imply the cubic rate result.[3] The latter is an implication of the quadratic rate result *and* the diminishing marginal benefit property. Intuitively, diminishing marginal benefit property ensures that the uncaptured profit $\Pi_\infty - \Pi_n$ would be captured by earlier extra varieties. Hence, the

[3]But the converse is true.

convergence rate of the marginal benefit $\Pi_{n+1}-\Pi_n$ would be faster than that of the uncaptured profit $\Pi_\infty - \Pi_n$. As yet another implication of cubic rate result and the diminishing marginal benefit property, the existence of a moderate marginal cost k of developing extra varieties (cost of complexity) can plausibly justify the optimal number of varieties (optimal complexity level) to be quite small. More precisely, the optimal number of varieties is of order no more than $1/k^{1/3}$.

Our fourth main result is what we call the "two-third result". It says that the monopolist can earn more than two-third of the unconstrained profit by offering only two varieties, i.e. $\Pi_2 > 2\Pi_\infty/3$.[4] The literature has results of this kind derived in the context of procurement and regulation, and matching (see below), but to the best of my knowledge, this is the first result of this kind in any nonlinear pricing-type context. Most, if not all, of this kind of results in the literature need to assume specific functional forms. The same applies to ours. For our two-third result to hold, we assume consumers' utility is linear and production cost is quadratic in quality (so called linear-quadratic model, an extensively studied one in the literature), and the distribution of virtual types satisfies a regularity condition. Once again, the diminishing marginal benefit property plays a major role in the proof.

The most related paper in the literature is the concurrently written one by Bergemann, Shen, Xu, and Yeh (2011). It proves the quadratic rate result in the context of nonlinear pricing. However, its analysis, which applies the quantization theory, works only for the linear-quadratic model. On the other hand, Wilson (1989) and Wilson (1993) also give quadratic rate results in the contexts that are mathematically different than ours, namely efficient rationing of services and Ramsey pricing respectively. But none of them provides a tight bound. They do not analyze marginal benefit of complicating the mechanism (which is crucial to the optimal choice of complexity level) and the performance of a simple mechanism relative to that of the second best or first best.

In the context of procurement contracting, Rogerson (2003) considers "Fixed Price Cost Reimbursement (FPCR) menus", that is, two-item menus where one item is a cost-reimbursement contract and the other item is a fixed-price contract, of which the principal allows the agent to pick one. He shows that, if the agent's utility is quadratic and the agent's type is distributed uniformly, then "the optimal FPCR menu always captures at least three-quarters of the gain that the optimal complex menu achieves". Chu and Sappington (2007) allow a more general family of power distributions, and show that a menu of two options, namely, a cost-reimbursement contract and a linear cost sharing contract, can always secure at least 73 percent of the gain. McAfee (2002) in the context of two-sided matching shows that, if matching surplus takes multiplicative form in agents' types and the distributions of types

[4] Of course, this, together with the diminishing marginal benefit property, implies $\Pi_1 > \Pi_\infty/3$.

satisfy some regularity conditions, a social planner can divide the agents of each side into only two classes such that the resulting "coarse matching" achieves at least half of the social gain achieved by the fully optimal matching. These results can be regarded as analogous to our two-third result.

My companion paper Wong (2012) considers nonlinear pricing and compares the maximum profits of different forms of pricing schemes (namely bundling, incremental discounts, and all-units discounts) with any common level of complexity. It is complementary to this paper for it sheds some light on how to choose among different forms of simple mechanisms.

Miravete (2007) uses a large sample of independent cellular telephone markets to structurally estimate a monopolistic nonlinear pricing model. His estimates suggests that "firms should only offer few tariff options if the product development costs of designing them are non-negligible." His finding empirically supports our theoretical results, and ours provides rationale for his.

The rest of the paper is organized as follows. Section 2 sets up the environment and provides the standard solution in the literature. Section 3 characterizes the optimal discrete offering and provides a preliminary analysis of the constrained profit conditional on the number of varieties. Section 4 presents the main results on properties of the constrained profit sequence. The proofs we do not provide in the text are in Appendix.

2 The Model

2.1 Environment

Consider the Mussa and Rosen (1978) monopolistic quality differentiation environment. A commodity can be produced by a monopolist in a spectrum of varieties. The hedonic attributes of the varieties are characterized by a one-dimensional nonnegative quality index q. The consumers have unit demand for the commodity, i.e. every consumer chooses to buy either 0 or 1 unit. A consumer who decides to buy the commodity must pick one of offered varieties. The higher the quality index of a variety, the higher the willingness-to-pay of a consumer is.

Consumers are heterogeneous in their types, which are indexed by t. The utility of a type t consumer who buys a variety with quality q and pays the price p is represented as $t \cdot v(q) - p$. If the consumer does not buy a unit, her utility is zero. The function $v : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable with the properties

$$v(0) = 0, \quad v'(q) > 0, \quad v''(q) \leq 0 \ \forall q \geq 0. \tag{1}$$

The distribution of t is characterized by a cumulative distribution function $F(\cdot)$, whose support is a compact interval $[\underline{t}, \bar{t}]$. We assume that $F(\cdot)$ admits a positive density function $f(\cdot)$ on the support. The type of every consumer is the consumer's private information. The monopolist only knows the prior distribution $F(\cdot)$.

The unit cost of producing variety with quality q is denoted as $c(q)$. The cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable with the properties

$$c(0) = 0, \quad c'(q) > 0 \;\forall q > 0, \quad c''(q) > 0 \;\forall q \geq 0. \tag{2}$$

We also assume

$$\frac{c'(0)}{v'(0)} < \bar{t} < \lim_{q \to \infty} \frac{c'(q)}{v'(q)} \tag{3}$$

to guarantee that the market is nontrivial and the optimal quality for each type is bounded. Note that for each type t, the efficient allocation of quality is either 0 if $tv'(0) < c'(0)$, or the $q \geq 0$ that solves $tv'(q) = c'(q)$ if $tv'(0) \geq c'(0)$. (Notice that not consuming a unit can be identified with consuming a variety with quality 0.)

2.2 Unconstrained solution

If the monopolist can costlessly establish as many varieties as it wants, the problem it faces is the standard problem studied by Mussa and Rosen (1978). That is, the monopolist solves

$$\max_{q(\cdot) \geq 0, p(\cdot)} \int_{\underline{t}}^{\bar{t}} [p(t) - c(q(t))] \, dF(t) \tag{4}$$

subject to

$$\begin{aligned} tv(q(t)) - p(t) &\geq 0 \qquad \forall t \in [\underline{t}, \bar{t}] & (5) \\ tv(q(t)) - p(t) &\geq tv(q(t')) - p(t') \qquad \forall t, t' \in [\underline{t}, \bar{t}]. & (6) \end{aligned}$$

The functions $p(t)$ and $q(t)$ specify the monopolist's choice of price and quality for consumers with type t. The objective function in (4) is the (per consumer) profit. The constraint (5) is the individual rationality (IR) constraint, which exists because every consumer has the outside option of buying nothing, paying nothing and getting the reservation utility zero. The constraint (6) is the incentive compatibility (IC) constraint, which arises from the fact that the consumers' types are private information.

Since the problem above has no constraint on the number of varieties to be offered (as opposed to the problems in later sections), and hence the standard solution in general involves a continuum of varieties, we call it the "unconstrained problem" or "unconstrained program", and the maximized value the "unconstrained profit", denoted as Π_∞.

Adopting the standard technique of solving this kind of problem[5], the unconstrained program is reduced to

$$\max_{q(\cdot)\geq 0} \int_{\underline{t}}^{\bar{t}} [J(t)v(q(t)) - c(q(t))]\, dF(t) \tag{7}$$

subject to the constraint that $q(t)$ is nondecreasing in t, where $J(t) \equiv t - \frac{1-F(t)}{f(t)}$ is the "virtual type function". Assume that $J(\cdot)$ is strictly increasing, which is a standard regularity condition in the literature. Then the monotonicity constraint of $q(\cdot)$ is not binding, and the profit-maximizing quality $q_\infty(t)$ for type t is such that

$$J(t)v'(q_\infty(t)) = c'(q_\infty(t)) \tag{8}$$

whenever type t is served, i.e. whenever $J(t)v'(0) \geq c'(0)$. For notational convenience, let us assume $J(\underline{t})v'(0) < c'(0)$ from now on, which means that some very low types of consumers will not be served.

Proposition 1 (Mussa and Rosen, 1978) *The unconstrained profit Π_∞ can be written as:*

$$\Pi_\infty = \int_{\underline{t}}^{\bar{t}} \max_{q\geq 0} \{J(t)v(q) - c(q)\}\, dF(t) = \int_{\underline{t}}^{\bar{t}} [J(t)v(q_\infty(t)) - c(q_\infty(t))]\, dF(t) \tag{9}$$

where the optimal continuous offering $q_\infty(\cdot)$ is defined by $q_\infty(t) = 0$ for $\underline{t} \leq t < t_*$ and by (8) for $t_* \leq t \leq \bar{t}$, where the lowest served type t_* is defined by $J(t_*)v'(0) = c'(0)$.

3 The Constrained Program

Now consider the situation where the monopolist restricts itself to offer at most n varieties. The IR, IC and nonnegativity constraints still remain. Therefore, the monopolist solves the following "constrained problem" or "constrained program":

$$\Pi_n = \max_{q(\cdot)\geq 0} \int_{\underline{t}}^{\bar{t}} [J(t)v(q(t)) - c(q(t))]\, dF(t) \tag{10}$$

subject to the constraints that $q(t)$ is nondecreasing in t, and

$$q(\cdot) \text{ takes at most } n \text{ values except zero.} \tag{11}$$

[5]See for example Fudenberg and Tirole (1991), Chapter 7.

It is nothing but the unconstrained program adding the constraint (11). We call Π_n the constrained profit and $\Pi_\infty - \Pi_n$ the uncaptured profit given the number n of varieties.

3.1 The optimal discrete offering

Now we are ready to analyze the optimal discrete offering in a constrained program, given the number of varieties n. First, it is not hard to show, under our assumptions, that the constrained program (10) has a solution, and the monotonicity constraint of $q(\cdot)$ is not binding. Moreover, from the monotonicity of $q(\cdot)$ and the constraint (11), $q(\cdot)$ must be a nondecreasing n-step function. Let q_i $(i = 1, \ldots, n)$ be the quality serving for the interval $[t_i, t_{i+1}]$ of types (with the convention that $t_{n+1} = \bar{t}$), we can rewrite the constrained program (10) as a $2n$-dimensional problem:

$$\max_{\substack{q_1,\ldots,q_n \in \mathbb{R}_+; \\ t_1,\ldots,t_n \in [\underline{t},\bar{t}]}} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} [J(t)v(q_i) - c(q_i)]\, dF(t). \tag{12}$$

The first-order necessary conditions of (12) with respect to $(q_1, \ldots, q_n; t_1, \ldots, t_n)$ can be written as two first-order difference equations

$$\int_{t_i}^{t_{i+1}} J(t)dF(t) \cdot v'(q_i) = c'(q_i) \cdot (F(t_{i+1}) - F(t_i)) \qquad \forall i = 1, \ldots, n, \tag{13}$$

$$J(t_i) \cdot (v(q_i) - v(q_{i-1})) = c(q_i) - c(q_{i-1}) \quad \forall i = 1, \ldots, n, \tag{14}$$

and two boundary conditions

$$q_0 = 0, \tag{15}$$

$$t_{n+1} = \bar{t}. \tag{16}$$

Therefore, the optimal discrete offering $(q_1, \ldots, q_n; t_1, \ldots, t_n)$ in a constrained program can be characterized as the solution of a system of two difference equations. This system does not have closed-form solution except for special cases (see Subsection 3.2). Not surprisingly, both the difference equations (13) and (14) converge to the unconstrained optimal offering formula $J(t)v'(q) = c'(q)$ as the consecutive t_i's and q_i's become closer and closer.

Proposition 2 *There exists a solution* $(q_1, \ldots, q_n; t_1, \ldots, t_n)$ *to the difference equations* (13) *and* (14), *coupled with the boundary conditions* (15) *and* (16), *such that*

$$\Pi_n = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} [J(t)v(q_i) - c(q_i)]\, dF(t). \tag{17}$$

Proof. First, in problem (10), the monotonicity constraint of $q(\cdot)$ is not binding. To see this, let the monotonicity constraint of $q(\cdot)$ be ignored for the moment, and suppose that some solution $q(\cdot)$ of this relaxed problem takes values $q_1, \ldots, q_n$ except zero, with $0 \leq q_1 \leq \cdots \leq q_n$. Then, since $q(\cdot)$ maximizes the objective function in (10), it must satisfy: for every $t \in [\underline{t}, \bar{t}]$, $q(t)$ is some q_i that maximizes the virtual surplus $J(t)\, v(q_i) - c(q_i)$ among $q_1, \ldots, q_n$. Now since $J(t)$ is increasing in t and $v(q)$ is increasing in q, the virtual surplus function $J(t)\, v(q) - c(q)$ satisfies strict increasing differences in (q, t), and hence $q(t)$ must be nondecreasing in t.[6]

Second, the equivalent version of the constrained program (12) has a solution, and hence Π_n is well-defined. To see this, notice that the objective function in (12) is continuous in $(q_1, \ldots, q_n; t_1, \ldots, t_n)$. Moreover, q_n can be without loss restricted to be below some large upper bound, because our assumption (3) implies that $\bar{t}v'(q) < c'(q)$ for all large enough q. Then the constraint set is compact, and from Weierstrass Theorem a maximizer exists.

Third, our previous analysis reveals that any solution to (12) satisfies (13), (14), (15) and (16). ■

The solution $q(\cdot)$ of the original constrained program (10) characterized by the above optimal discrete offering $(q_1, \ldots, q_n; t_1, \ldots, t_n)$ is denoted as $q_n(\cdot)$, which we also call optimal discrete offering. From condition (13), as long as $t_i < t_{i+1}$ (which must be the case from the proof of Proposition 3(i) below), we must have

$$\frac{c'(q_\infty(t_i))}{v'(q_\infty(t_i))} = J(t_i) < \frac{c'(q_i)}{v'(q_i)} < J(t_{i+1}) = \frac{c'(q_\infty(t_{i+1}))}{v'(q_\infty(t_{i+1}))},$$

and hence

$$q_\infty(t_i) < q_i < q_\infty(t_{i+1}).$$

The pattern of an optimal discrete offering is sketched in Figure 1: the optimal discrete offering of a constrained program is a step-function approximation of the optimal continuous offering of the unconstrained program.[7]

Once the solution of $(q_1, \ldots, q_n; t_1, \ldots, t_n)$ is characterized, the solution of $(p_1, \ldots, p_n)$ easily follows. Namely,

$$p_1 = t_1 q_1,$$

$$p_i = p_{i-1} + t_i(q_i - q_{i-1}) \quad \forall i = 2, \ldots, n.$$

The first one is from the fact that a type t_1 consumer will be indifferent between buying the

[6] See the monotone comparative statics results in Milgrom and Shannon (1994).

[7] Even if the virtual type function J is not monotone so that the monotonicity of $q(\cdot)$ might be binding, (13) – (16) are still necessary conditions for the constrained program. Hence our analysis is still valid.

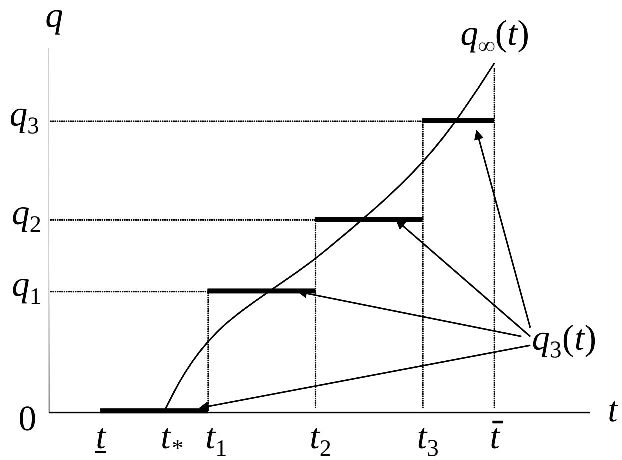

Figure 1: Comparison between the optimal continuous offering $q_\infty(\cdot)$ and the optimal discrete offering $q_3(\cdot)$ when the number of varieties is 3

variety with quality q_1 or buying nothing. The second one is due to the fact that a type t_i consumer will be indifferent between buying the variety with quality q_i or with quality q_{i-1}.

3.2 A solvable class of examples

This subsection studies a class of environments which one may call "linear-quadratic-uniform" cases. In these cases the optimal discrete offerings and the constrained profits can be explicitly solved, revealing a number of features that we will generalize to our general setup.

In this subsection, assume that the distribution of consumers' types is uniform on the support $[\underline{t}, \bar{t}]$, which implies that $J(t) = 2t - \bar{t}$. Also assume that consumers' utility is linear in quality and the seller's unit cost of production is quadratic in quality, i.e.

$$v(q) = A_0 q, \quad c(q) = A_1 q + \frac{A_2}{2} q^2 \tag{18}$$

where A_0 and A_2 are positive constants and A_1 is a nonnegative constant. Our assumptions (3) and $J(\underline{t})v'(0) < c'(0)$ reduce to $(2\underline{t} - \bar{t})\, A_0 < A_1 < \bar{t}A_0$. The optimal continuous offering is $q_\infty(t) = ((2t - \bar{t})A_0 - A_1)/A_2$ for $t \geq t_*$, where $t_* = (\bar{t}A_0 + A_1)/2A_0$. Substituting into (9) and simplifying, we have

$$\Pi_\infty = \frac{(\bar{t}A_0 - A_1)^3}{12 A_0 A_2 (\bar{t} - \underline{t})}. \tag{19}$$

Since $J(\cdot)$ and $c'(\cdot)$ are linear and $v'(\cdot) = A_0$ is a constant, the first-order difference equations (13) and (14) become linear as follows:

$$\frac{J(t_{i+1})A_0 + J(t_i)A_0}{2} = c'(q_i) \quad \forall i = 1, \ldots, n, \tag{20}$$

$$J(t_i)A_0 = \frac{c'(q_i) + c'(q_{i-1})}{2} \quad \forall i = 1, \ldots, n. \tag{21}$$

With the two boundary conditions (15) and (16), it is easy to see that the optimal discrete offering satisfies

$$J(t_i)A_0 = c'(0) + \frac{2i-1}{2n+1}\left(\bar{t}A_0 - c'(0)\right),$$

$$c'(q_i) = c'(0) + \frac{2i}{2n+1}\left(\bar{t}A_0 - c'(0)\right),$$

or equivalently,

$$q_i = \frac{2i}{2n+1} \cdot \frac{\bar{t}A_0 - A_1}{A_2}, \tag{22}$$

$$t_i = \frac{A_1}{A_0} + \frac{n+i}{2n+1} \cdot \frac{\bar{t}A_0 - A_1}{A_0}. \tag{23}$$

Substituting into the Π_n expression (17) and simplifying, we have

$$\Pi_n = \frac{4n(n+1)}{(2n+1)^2} \cdot \frac{(\bar{t}A_0 - A_1)^3}{12A_0A_2(\bar{t} - \underline{t})} = \frac{4n(n+1)}{(2n+1)^2}\Pi_\infty. \tag{24}$$

Thus, Π_n is monotonically increasing to Π_∞ as $n \to \infty$; and the uncaptured profit

$$\Pi_\infty - \Pi_n = \frac{1}{(2n+1)^2} \cdot \frac{(\bar{t}A_0 - A_1)^3}{12A_0A_2(\bar{t} - \underline{t})} = \frac{1}{(2n+1)^2}\Pi_\infty \tag{25}$$

is of order $1/n^2$. The fraction of constrained profit out of the unconstrained profit Π_n/Π_∞ is approximately 89%, 96% and 98% for $n = 1, 2$ and 3 respectively. The marginal benefit of adding one more variety

$$\Pi_{n+1} - \Pi_n = \frac{8(n+1)}{(2n+1)^2(2n+3)^2}\Pi_\infty$$

is monotonically decreasing to 0, and is of order $1/n^3$. The associated percentage change

$$\frac{\Pi_{n+1} - \Pi_n}{\Pi_n} = \frac{2}{n(2n+3)^2}$$

is approximately 8%, 2% and 0.8% for $n = 1, 2$ and 3 respectively.[8]

[8] The 2009 version of this paper also contains numerical simulations for other examples. In each of these

3.3 Visualizations of constrained and unconstrained profit

We come back to our general setup. In order to nicely visualize the constrained and unconstrained profits and then prove our results, it is convenient to use a change of variable $x \equiv J(t)$, because the integrand in the constrained and unconstrained profits is linear in virtual type $J(t)$. Define

$$H(q,x) \equiv x \cdot v(q) - c(q), \tag{26}$$

$$G(x) \equiv F(J^{-1}(x)), \quad x_* \equiv J(t_*) = \frac{c'(0)}{v'(0)}.$$

$H(q,x)$ is the virtual surplus at quality q and virtual type x; $G(\cdot)$ is the distribution of virtual types. Now the unconstrained profit can be written as

$$\Pi_\infty = \int_{x_*}^{\bar{t}} \max_{q\geq 0} H(q,x)\, dG(x) = \int_{x_*}^{\bar{t}} H\left(q_\infty(J^{-1}(x)), x\right) dG(x). \tag{27}$$

Given n and the corresponding optimal discrete offering $(q_1,\ldots,q_n;t_1,\ldots,t_n)$, the constrained profit can be written as

$$\Pi_n = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} H(q_i,x)\, dG(x), \tag{28}$$

where $x_i \equiv J(t_i)$ for $i = 1,\ldots,n$, and $x_{n+1} \equiv \bar{t}$.

Equation (14) can be written as

$$H(q_i,x_i) = H(q_{i-1},x_i) \quad \forall i = 1,\cdots,n. \tag{29}$$

Fixing any q_i, the slope of $H(q_i,x)$ with respect to x is

$$\frac{\partial H(q_i,x)}{\partial x} = v(q_i) > 0.$$

Now, we can nicely visualize Π_∞ and Π_n in a single diagram.[9] First note that for any i, the two curves $H(q_i,x)$ and $H(q_{i-1},x)$ plotted against x must cross only once at $x = x_i$, because $H_{12} > 0$. When plotted against $G(x)$, they must cross only once at $G(x) = G(x_i)$. Moreover, the curve $\max_{q\geq 0} H(q,x)$ plotted against x or against $G(x)$ is the upper envelope of all the curves $H(q,x)$ with various values of q. The ideas are shown by Figure 2, in which $n = 3$. From (27), it is clear that Π_∞ is the area below the bold curve $\max_{q\geq 0} H(q,x)$ in Figure 2.

examples Π_3/Π_∞ is more than 97% and $(\Pi_4 - \Pi_3)/\Pi_3$ is less than 1%. See Wong (2009).

[9] Analyzing the social planner's problems or the perfect information monopolist problems (both constrained and unconstrained) only amounts to replacing $J(t)$ by t, or replacing $G(\cdot)$ by $F(\cdot)$. All our results can be easily adapted there.

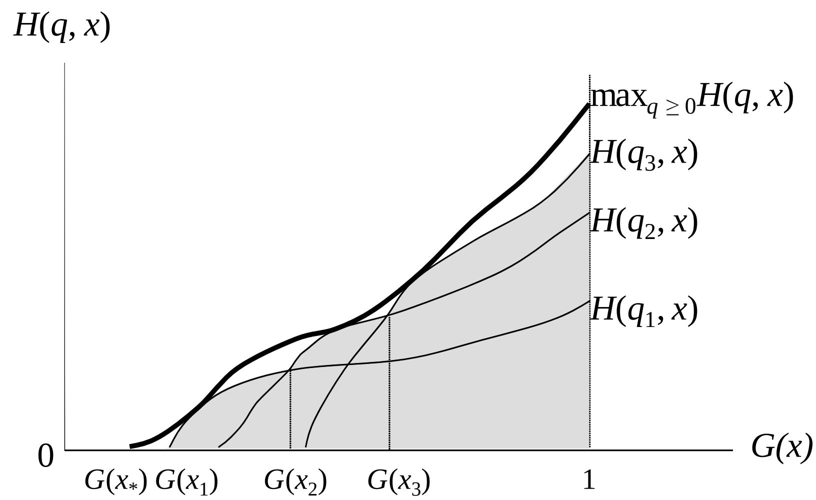

Figure 2: Visualizations of the unconstrained profit Π_∞ and the constrained profit Π_n

Moreover, (28) and (29) say that Π_n is represented as the shaded area.

An important insight shed from Figure 2 is that each of the varieties offered helps capturing the unconstrained profit Π_∞ in a first-order sense, since the slope of $H(q,x)$ with respect to quality q is flat at the ideal second best quality. The uncaptured profit is therefore of second or higher order. We will use this idea and apply Taylor's Theorem to show that the uncaptured profit is of order no more than $1/n^2$ in Subsection 4.2.

4 Properties of the Constrained Profit Sequence

4.1 Basic properties and diminishing marginal benefit

The sequence of constrained profit $\{\Pi_n\}_{n=0}^{\infty}$ has the following properties. (Notice that Π_0 is also well defined: when $n = 0$, no variety can be offered, so that $q_0(t) = 0$ and $\Pi_0 = 0$. But of course, the analyses in Subsections 3.1 and 3.3 are only for $n \geq 1$.)

Proposition 3 *(i)* $\Pi_n > \Pi_{n-1}$ *for every* $n = 1, 2, \ldots$*; (ii)* $\lim\limits_{n\to\infty} \Pi_n = \Pi_\infty$*, where* Π_∞ *is characterized in Proposition 1; and (iii)* $\lim\limits_{n\to\infty} (\Pi_n - \Pi_{n-1}) = 0.$

Proof. Obviously, $\Pi_1 > 0 = \Pi_0$. So suppose $n \geq 2$. An increase in n (i.e. more varieties allowed) weakens constraint (11) of the constrained program, thus $\Pi_n \geq \Pi_{n-1}$ for every n. Moreover, this inequality must be strict because the optimal discrete offering in a n-variety program must involve n different varieties. To see this, suppose $q(\cdot)$ takes only $n-1$ values except zero in a n-variety program. Because the optimal continuous offering $q_\infty(\cdot)$ (defined in Proposition 1) is strictly increasing, some $\hat{q}(\cdot)$ function that takes n values except zero can approximate $q_\infty(\cdot)$ better than $q(\cdot)$ does, i.e. $|q_\infty(t) - \hat{q}(t)| \leq |q_\infty(t) - q(t)|$ for all $t \in [\underline{t}, \bar{t}]$ and $|q_\infty(t) - \hat{q}(t)| < |q_\infty(t) - q(t)|$ for all t in some open subset of $[\underline{t}, \bar{t}]$. Since the objective function's integrand $J(t)\, v(q) - c(q)$ is single-peaked in q with unique maximizer $q_\infty(t)$, we see that the n-variety offering $\hat{q}(\cdot)$ yields strictly higher profit than the $(n-1)$-variety offering $q(\cdot)$ does. It proves (i).

Now the sequence $\{\Pi_n\}_{n=0}^{\infty}$ is increasing and bounded from above by Π_∞. Thus it has a finite limit, and $\lim_{n\to\infty} \Pi_n \leq \Pi_\infty$. To see the equality in (ii), notice that, as an increasing function, $q_\infty(\cdot)$ can be arbitrarily well approximated by nondecreasing finite-step functions.

Finally, since any convergent sequence in Euclidean space is also a Cauchy sequence, $(\Pi_n - \Pi_{n-1})$ converges to zero as n goes to infinity. It proves (iii). ■

We also prove that the marginal constrained profit $\Pi_n - \Pi_{n-1}$ is strictly decreasing in n. Intuitively, as the number of varieties that have already been offered is larger, the space for improving profit by adding an extra variety is smaller, and hence the effectiveness of the extra variety is less. However, this "diminishing marginal benefit" property is far from trivial, because adding one more variety would bring about an optimal adjustment of all previously offered varieties. Although $\Pi_n - \Pi_{n-1}$ must ultimately diminish, it is rather surprising that the property holds for every n. This diminishing marginal benefit property is not only interesting on its own, but also crucial to our analysis of the rates of convergence of this marginal benefit and the monopoly choice of varieties, and our result on Π_2 in linear-quadratic model.

Theorem 1 (Diminishing marginal benefit) *The increments of* $\{\Pi_n\}_{n=0}^{\infty}$ *are decreasing, i.e.* $\Pi_{n+1} - \Pi_n < \Pi_n - \Pi_{n-1}$ *for every* $n = 1, 2, \ldots$.

The proof of Theorem 1 is in Appendix. To understand the idea behind the proof, let us sketch the proof for the first two inequalities: $\Pi_2 - \Pi_1 < \Pi_1 - \Pi_0$ and $\Pi_3 - \Pi_2 < \Pi_2 - \Pi_1$. The first one is equivalent to $\Pi_2 < 2\Pi_1$. The left panel of Figure 3 visualizes Π_2. Let $q_{1,2}$ and $q_{2,2}$ be the two quality levels involved in the optimal discrete offering when $n = 2$. Now imagine two plans of the monopolist: the first plan is to offer only one variety with quality $q_{1,2}$, while the second plan is to offer only one variety with quality $q_{2,2}$. The total profit from these two plans is visualized in the right panel of Figure 3, where a "2" in an area indicates that that area should be counted twice because that area accounts for profits from both plans, while a "1" in an area indicates that that area should be counted once because that area accounts for profit from only one of the two plans. Now $2\Pi_1$ must be greater than the total profit from the two plans, and from Figure 3 it is obvious that the two plans make a total profit greater than Π_2. It proves that $\Pi_2 < 2\Pi_1$.

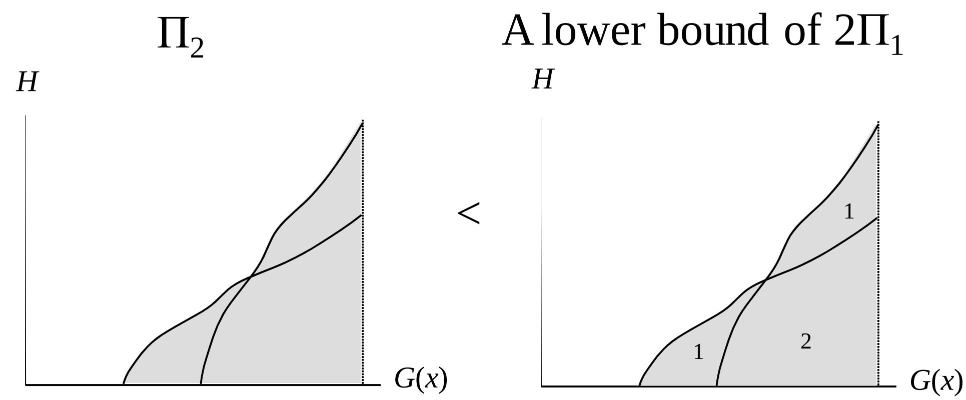

Figure 3: Comparing Π_2 and $2\Pi_1$

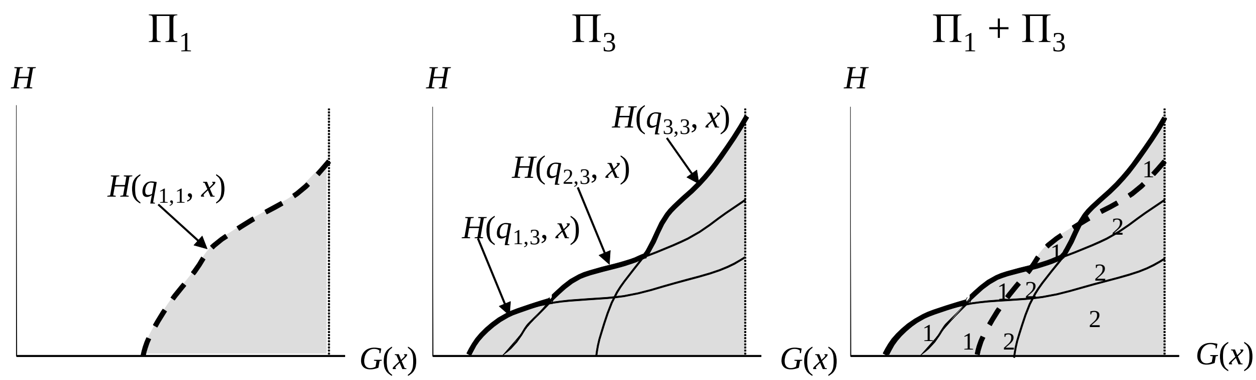

Figure 4: Visualization of $\Pi_1 + \Pi_3$

The second inequality, which is equivalent to $\Pi_1 + \Pi_3 < 2\Pi_2$, is harder. The shaded areas in the left and middle panels of Figure 4 visualize Π_1 and Π_3 respectively. The right panel of Figure 4 visualizes the sum $\Pi_1 + \Pi_3$. In this right panel, Π_1 is the area below the dashed curve, and Π_3 is the area below the bold solid curve. A "2" in an area indicates that that area should be counted twice because that area occurs in both Π_1 and Π_3. Similarly, a "1" in an area indicates that that area should be counted only once because that area occurs in either Π_1 or Π_3, not both. Next I will show that $2\Pi_2$ must be greater than the area indicating $\Pi_1 + \Pi_3$. To do this, we only need to construct two (suboptimal) 2-variety menus such that the sum of those two corresponding (suboptimal) 2-variety profits is larger than $\Pi_1 + \Pi_3$. The following procedure will do. First, let the optimal quality involved when $n = 1$ is $q_{1,1}$ and the optimal qualities involved when $n = 3$ are $q_{1,3}$, $q_{2,3}$ and $q_{3,3}$. Rank all the qualities involved in the above two constrained programs. For the example shown in Figure 4, this ranking is $q_{1,3} < q_{2,3} < q_{1,1} < q_{3,3}$. Then, collect those with odd ranking in

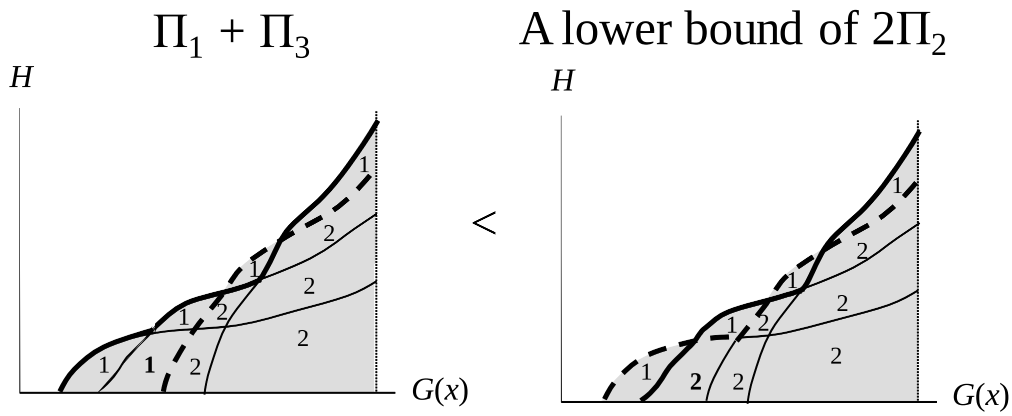

Figure 5: Comparing $\Pi_1 + \Pi_3$ and $2\Pi_2$

one menu and those with even ranking in another menu. For the current example, the two menus are $(q_{1,3}, q_{1,1})$ and $(q_{2,3}, q_{3,3})$. Whereas $\Pi_1 + \Pi_3$ is visualized in the left panel of Figure 5, the sum of the two 2-variety profits is visualized in the right panel of Figure 5. It is now easy to see that $\Pi_1 + \Pi_3 < 2\Pi_2$ because the two panels of Figure 5 are the same except that a "1" in the left panel is replaced by a "2" in the right panel.

The above logic is valid in general, and it is the intuition behind the proof of Theorem 1.

Corollary 1 *For every $n = 1, 2, \ldots,$*

$$\frac{\Pi_n - \Pi_{n-1}}{\Pi_n} \leq \frac{1}{n}.$$

Proof. Theorem 1 implies that, for every $n = 1, 2, \ldots,$

$$\Pi_n = (\Pi_n - \Pi_{n-1}) + (\Pi_{n-1} - \Pi_{n-2}) + \cdots (\Pi_1 - \Pi_0) \geq n (\Pi_n - \Pi_{n-1}).$$

■

4.2 Rate of convergence results

This subsection provides our rate of convergence results for uncaptured profit $\Pi_\infty - \Pi_n$, marginal profit $\Pi_{n+1} - \Pi_n$, and monopoly choice of the number of varieties.

Theorem 2 (Quadratic rate result) *There exists a finite constant $M_0 < \infty$, not depending on n, such that for every $n = 0, 1, 2, \ldots,$*

$$\Pi_\infty - \Pi_n \leq \frac{M_0}{(2n+1)^2}. \tag{30}$$

Thus, $\Pi_\infty - \Pi_n = O(1/n^2)$*. Moreover, if the distribution* $G(\cdot)$ *of virtual types, given by* $G(J(t)) = F(t)$*, admits a density* $g(\cdot)$ *over the support* $[J(t_*), \bar{t}]$ *that is bounded from above by* $\bar{g}$*, then a tight bound is given by*

$$M_0 = \frac{M_1 \bar{g}}{6} \cdot \left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^3, \tag{31}$$

where

$$M_1 \equiv \sup_{q \in [0, q(\bar{t})]} \left\{ \frac{(v'(q))^3}{v'(q)c''(q) - c'(q)v''(q)} \right\} < \infty.$$

Thus, the uncaptured profit $\Pi_\infty - \Pi_n$ is of order no more than $1/n^2$. The results for linear-quadratic-uniform model in Subsection 3.2 show that this convergence rate is tight. The bound given by (30) and (31) is also tight. To see this, observe that under the linear-quadratic-uniform model in Subsection 3.2, M_1 is A_0^2/A_2 and $\bar{g}$ is $1/2(\bar{t} - \underline{t})$, so that the bound of $\Pi_\infty - \Pi_n$ given by Theorem 2 is exactly the same as the explicit solution of $\Pi_\infty - \Pi_n$ given by (25).

The proof of Theorem 2 is in Appendix. The underlying reason of this quadratic rate result has already been explained in Introduction and Subsection 3.3. The main point is that the slope of the virtual surplus function $J(t)v(q) - c(q)$ with respect to quality q is flat at its maximizer $q_\infty(t)$. A Taylor expansion argument shows that the uncaptured profit is of order no more than $1/n^2$ as long as the virtual surplus function is smooth enough in q. The smoothness condition we need is precisely the twice continuous differentiability of the value function $v(\cdot)$ and cost function $c(\cdot)$, which is a standard one in the literature. Moreover, as can be seen from the proof, this convergence rate of Π_n can be attained by a simple offering rule, which only involves uniformly distributed set of (suboptimal) varieties.

Theorem 3 (Cubic rate result) *For every* $n = 1, 2, \ldots,$

$$\Pi_{n+1} - \Pi_n < \frac{27}{8n^2(2n+9)} M_0,$$

where M_0 *is a bound of* $(2n+1)(\Pi_\infty - \Pi_n)$*, which exists from Theorem 2. Thus,* $\Pi_{n+1} - \Pi_n = O(1/n^3)$*.*

Proof. For any positive integers n, i with $n \geq i \geq 1$, we have

$$\Pi_\infty - \Pi_i = (\Pi_\infty - \Pi_{n+1}) + (\Pi_{n+1} - \Pi_n) + \cdots + (\Pi_{i+1} - \Pi_i). \tag{32}$$

By Theorem 2, the left-hand side of (32) is bounded by

$$\Pi_\infty - \Pi_i \leq \frac{M_0}{(2i+1)^2}.$$

The right-hand side of (32) is bounded by

$$(\Pi_\infty - \Pi_{n+1}) + (\Pi_{n+1} - \Pi_n) + \cdots + (\Pi_{i+1} - \Pi_i) > (n-i+1)(\Pi_{n+1} - \Pi_n),$$

due to Theorem 1 and the fact that $\Pi_{n+1} < \Pi_\infty$. Combining the above results, we have

$$\Pi_{n+1} - \Pi_n < \frac{M_0}{(n-i+1)(2i+1)^2} \quad \forall i = 1, \ldots, n.$$

It remains to find a tight lower bound for

$$\max_{i \in \{1,\ldots,n\}} (n-i+1)(2i+1)^2.$$

Since the unique maximizer of $(n-i+1)(2i+1)^2$ on $\mathbb{R}_+$ is $(4n+3)/6$, let us define $i^*(n)$ as the largest integer that does not exceed $(4n+3)/6$. Then

$$\frac{4n-3}{6} < i^*(n) \leq \frac{4n+3}{6},$$

$$\begin{aligned} \max_{i \in \{1,\ldots,n\}} (n-i+1)(2i+1)^2 &\geq (n - i^*(n) + 1)(2i^*(n)+1)^2 \\ &> \left(n - \frac{4n-3}{6} + 1\right)\left(2\left(\frac{4n-3}{6}\right) + 1\right)^2 \\ &= \frac{8}{27} n^2 (2n+9). \end{aligned}$$

Therefore,

$$\Pi_{n+1} - \Pi_n < \frac{M_0}{\max_{i \in \{1,\ldots,n\}} (n-i+1)(2i+1)^2} < \frac{27}{8n^2(2n+9)} M_0.$$

■

Theorem 1 and Theorem 3 together tell us that the marginal constrained profit $\Pi_n - \Pi_{n-1}$ monotonically converges to its limit zero at the cubic rate.

If there is a (per consumer) fixed cost $k > 0$ of developing each variety, and the monopolist can freely choose the number of varieties, then the optimal number of varieties n^* is the maximizer of the profit net of the cost for developing varieties. As the per person cost k of developing a variety goes to zero, the optimal number of varieties $n^*(k)$ certainly goes to infinity. But the following corollary of Theorem 1 and Theorem 3 tells us that $n^*(k)$ goes to infinity at an extremely slow rate, in particular, it is of order no more than $1/k^{1/3}$.

Corollary 2 *Let $n^*(k)$ be an (usually unique) optimal number of varieties when the per consumer cost of developing every variety is $k > 0$, i.e. $n^*(k)$ maximizes $\Pi_n - nk$ over $n \in \{0, 1, 2, \ldots\}$. Then, $n^*(k) = O(1/k^{1/3})$ as $k \to 0$.*

Proof. If k is too large, then $n^*(k) = 0$. Suppose $k > 0$ is not so large, such that $n^*(k) > 0$. Since $\Pi_{n+1} - \Pi_n$ is decreasing in n due to Theorem 1, $n^*(k)$ is characterized by $\Pi_{n^*(k)+1} - \Pi_{n^*(k)} \leq k$ and $\Pi_{n^*(k)} - \Pi_{n^*(k)-1} \geq k$. It follows from Theorem 3 that

$$k^{1/3} n^*(k) \leq \left(\Pi_{n^*(k)} - \Pi_{n^*(k)-1}\right)^{1/3} n^*(k) = O(1).$$

■

4.3 Performance of 2-variety menus in linear-quadratic model

In this subsection we make more assumptions. First, consumers' utility is linear in quality and the seller's unit cost of production is quadratic in quality. That is, the functions $v(\cdot)$ and $c(\cdot)$ are specified as in (18). Second, the distribution $G(\cdot)$ of virtual types, given by $G(J(t)) \equiv F(t)$, has a positive density $g(\cdot)$ on the support $[x_*, \bar{t}]$, where $x_* \equiv J(t_*) = c'(0)/v'(0)$. Third, the distribution $G(\cdot)$ satisfies the following regularity condition:

$$\frac{G(x) - G(x_*)}{g(x)} \text{ and } -\frac{1 - G(x)}{g(x)} \text{ are nondecreasing in } x \text{ on } [x_*, \bar{t}].$$ [10]

Consider a auxiliary problem with, first, the types below t_* (where $J(t_*)A_0 = A_1$) are thrown out, and second, all types above t_* has to be covered (i.e. the outside option of buying nothing is not available). For any positive integer n or $n = \infty$, the maximum profit is then[11]

$$\begin{aligned}
\hat{\Pi}_n &= \max_{q(\cdot)\geq 0} \int_{t_*}^{\bar{t}} \left[J(t)v(q(t)) - c(q(t))\right] dF(t) \\
&= \max_{q(\cdot)\geq 0} \int_{t_*}^{\bar{t}} \left[\left(J(t)A_0 - A_1\right) q(t) - \frac{A_2}{2}\left(q(t)\right)^2\right] dF(t)
\end{aligned}$$

[10] Equivalently, it says $\frac{F(t)}{f(t)}J'(t)$ and $-\frac{1-F(t)}{f(t)}J'(t)$ are nondecreasing in t on $[t_*, \bar{t}]$.

[11] Unlike our original constrained program, the integrand in $\hat{\Pi}_n$ evaluated at the optimal discrete offering is typically negative for very low types.

subject to

$$q(\cdot) \text{ takes at most } n \text{ values (not except zero).}$$

Also define $\hat{\Pi}_0 \equiv 0$.

There are two remarks to make. First, $\hat{\Pi}_n < \Pi_n$ for any $n \notin \{0, \infty\}$, and $\hat{\Pi}_\infty = \Pi_\infty$. Second, it can be easily seen that all of our previous results for $\{\Pi_n\}_{n=0}^\infty$ also hold for $\{\hat{\Pi}_n\}_{n=0}^\infty$. In particular, the diminishing marginal benefit result can be proved in exactly the same way. (Of course, the bounds in our rate of convergence results need to be modified. For example, the factor $(2n+1)^2$ in Theorem 2 should be replaced by $(2n)^2$.)

Lemma 1 *Considering the auxiliary problem, the benefit of going from offering one variety to two varieties accounts for at least half of the total benefit of going all the way to the optimal continuous offering, i.e.*

$$\hat{\Pi}_2 - \hat{\Pi}_1 \geq \Pi_\infty - \hat{\Pi}_2.$$

The proof of Lemma 1 is in Appendix, which uses the technique in McAfee (2002). Basically, the proof considers a suboptimal 2-variety menu with one variety serving those consumers with below-mean virtual types, and another variety serving those consumers with above-mean virtual types. It can be shown that the sum of Π_∞ and $\hat{\Pi}_1$, which have explicit forms under linear-quadratic model, cannot be larger than two times of the profit attained by the above suboptimal 2-variety menu.

Lemma 1, together with diminishing marginal benefit, implies that offering only two varieties can capture more than two-third of the unconstrained maximum profit.

Theorem 4 (Two-third result)

$$\Pi_2 > \hat{\Pi}_2 > \frac{2}{3}\Pi_\infty.$$

Proof. As noted before, $\{\hat{\Pi}_n\}_{n=0}^\infty$ has the diminishing marginal benefit property. In particular, $\hat{\Pi}_2 < 2\hat{\Pi}_1$. It, together with Lemma 1, implies

$$\begin{aligned} \Pi_\infty &= \left(\Pi_\infty - \hat{\Pi}_2\right) + \left(\hat{\Pi}_2 - \hat{\Pi}_1\right) + \hat{\Pi}_1 \\ &\leq 2\left(\hat{\Pi}_2 - \hat{\Pi}_1\right) + \hat{\Pi}_1 < 3\hat{\Pi}_1. \end{aligned}$$

Thus, $\hat{\Pi}_1 > \Pi_\infty/3$, and hence

$$\Pi_\infty - \hat{\Pi}_2 \leq \hat{\Pi}_2 - \hat{\Pi}_1 < \hat{\Pi}_2 - \Pi_\infty/3$$

$$2\hat{\Pi}_2 > 4\Pi_\infty/3.$$

■

Appendix

In order to prove Theorem 1, we firstly introduce a simple lemma.

Lemma 2 *Let S_1 and S_2 be two vectors of real numbers. (Their dimensions could be different.) Then*

$$\max_{(2)}(S_1, S_2) \geq \min\{\max(S_1), \max(S_2)\},$$

where $\max_{(2)}(S)$ denotes the second largest element in S.

Proof. If $\max(S_1, S_2)$ is in S_1, then $\max_{(2)}(S_1, S_2) \geq \max(S_2)$. If $\max(S_1, S_2)$ is in S_2, then $\max_{(2)}(S_1, S_2) \geq \max(S_1)$. In both cases, our claim is true. ■

Proof of Theorem 1. Let the optimal discrete offering given any number n of varieties be $(q_{1,n}, \ldots, q_{n,n}; t_{1,n}, \ldots, t_{n,n})$, and let $x_{i,n} \equiv J(t_{i,n})$. Notice that, for all $i, m \in \{1, 2, \ldots, n\}$ and all $x \in [x_{i,n}, x_{i+1,n}]$, we have $H(q_{i,n}, x) \geq H(q_{m,n}, x)$ (see Figure 2). Therefore, it follows from (28) that, for any n,

$$\Pi_n = \int_{x_*}^{\bar{t}} \max\left(0, H(q_{1,n}, x), H(q_{2,n}, x), \ldots, H(q_{n,n}, x)\right) dG(x).$$

Now, for every x and n define the vector $S_n(x)$ as

$$S_n(x) \equiv \left(0, H(q_{1,n}, x), H(q_{2,n}, x), \ldots, H(q_{n,n}, x)\right).$$

Then

$$\Pi_{n-1} = \int_{x_*}^{\bar{t}} \max(S_{n-1}(x))dG(x).$$

Similarly,

$$\Pi_{n+1} = \int_{x_*}^{\bar{t}} \max(S_{n+1}(x))dG(x).$$

Now, let us construct two suboptimal n-variety menus in the following way. First order the $2n$ numbers $q_{1,n-1}, q_{2,n-1}, \ldots, q_{n-1,n-1}, q_{1,n+1}, q_{2,n+1}, \ldots, q_{n+1,n+1}$ ascendingly and denote the ordered numbers as $q_{(1)}, q_{(2)}, \ldots, q_{(2n)}$, with the convention that $q_{(i)}$ is the i-th smallest number. Let us also define $q_{(0)}$ as 0.

Construct the first menu to include $q_{(1)}, q_{(3)}, q_{(5)}, \ldots, q_{(2n-1)}$ and the second menu to include $q_{(2)}, q_{(4)}, q_{(6)}, \ldots, q_{(2n)}$. And the corresponding x_i's are constructed optimally given the quality offers, so that the corresponding profits $\hat{\Pi}_n^{odd}$ and $\hat{\Pi}_n^{even}$ for these two menus are

$$\hat{\Pi}_n^{odd} = \int_{x_*}^{\bar{t}} \max\left(0, H(q_{(1)}, x), H(q_{(3)}, x), \ldots, H(q_{(2n-1)}, x)\right) dG(x),$$

$$\hat{\Pi}_n^{even} = \int_{x_*}^{\bar{t}} \max\left(0, H(q_{(2)}, x), H(q_{(4)}, x), \ldots, H(q_{(2n)}, x)\right) dG(x).$$

Notice that for any x, if

$$\max\left(0, 0, H(q_{(1)}, x), H(q_{(2)}, x), H(q_{(3)}, x), \ldots, H(q_{(2n)}, x)\right)$$

is $H(q_{(i)}, x)$, then the corresponding second largest element

$$\max_{(2)}\left(0, 0, H(q_{(1)}, x), H(q_{(2)}, x), H(q_{(3)}, x), \ldots, H(q_{(2n)}, x)\right)$$

must be either $H(q_{(i-1)}, x)$ or $H(q_{(i+1)}, x)$ from that fact that $H(q, x)$ satisfies increasing differences (see Figure 2). Therefore,

$$\begin{aligned}
\hat{\Pi}_n^{odd} + \hat{\Pi}_n^{even} &= \int_{x_*}^{\bar{t}} \Big[\max\left(0, 0, H(q_{(1)}, x), H(q_{(2)}, x), H(q_{(3)}, x), \ldots, H(q_{(2n)}, x)\right) \\
&\quad + \max_{(2)}\left(0, 0, H(q_{(1)}, x), H(q_{(2)}, x), H(q_{(3)}, x), \ldots, H(q_{(2n)}, x)\right)\Big] dG(x) \\
&= \int_{x_*}^{\bar{t}} \max(S_{n-1}(x), S_{n+1}(x)) dG(x) + \int_{x_*}^{\bar{t}} \max_{(2)}(S_{n-1}(x), S_{n+1}(x)) dG(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Pi_{n-1} + \Pi_{n+1} &= \int_{x_*}^{\bar{t}} [\max(S_{n-1}(x)) + \max(S_{n+1}(x))] dG(x) \\
&= \int_{x_*}^{\bar{t}} [\max\{\max(S_{n-1}(x)), \max(S_{n+1}(x))\} \\
&\quad + \min\{\max(S_{n-1}(x)), \max(S_{n+1}(x))\}] dG(x) \\
&= \int_{x_*}^{\bar{t}} [\max(S_{n-1}(x), S_{n+1}(x)) + \min\{\max(S_{n-1}(x)), \max(S_{n+1}(x))\}] dG(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left(\hat{\Pi}_n^{odd} + \hat{\Pi}_n^{even}\right) - (\Pi_{n-1} + \Pi_{n+1}) \\
= & \int_{x_*}^{\bar{t}} [\max_{(2)}(S_{n-1}(x), S_{n+1}(x)) - \min\{\max(S_{n-1}(x)), \max(S_{n+1}(x))\}] dG(x) \\
\geq & \ 0.
\end{aligned}$$

The last inequality is from Lemma 2.

Now, we have

$$2\Pi_n - (\Pi_{n-1} + \Pi_{n+1}) \geq \left(\hat{\Pi}_n^{odd} + \hat{\Pi}_n^{even}\right) - (\Pi_{n-1} + \Pi_{n+1}) \geq 0, \tag{33}$$

which implies $\Pi_n - \Pi_{n-1} \geq \Pi_{n+1} - \Pi_n$.

It remains to show that the two inequalities in (33) cannot be both equality. The second inequality in (33) is strict unless

$$\max_{(2)}(S_{n-1}(x), S_{n+1}(x)) = \min\{\max(S_{n-1}(x)), \max(S_{n+1}(x))\} \text{ for almost all } x.$$

By the definitions of $S_{n-1}(x)$ and $S_{n+1}(x)$ and noticing that $S_{n+1}(x)$ has two more elements than $S_{n-1}(x)$, the above cannot hold unless

$$\begin{aligned}
q_{1,n-1} &= q_{2,n+1} \\
q_{2,n-1} &= q_{3,n+1} \\
&\vdots \\
q_{n-2,n-1} &= q_{n-1,n+1} \\
q_{n-1,n-1} &= q_{n,n+1}.
\end{aligned}$$

But given the above, the constructed varieties for $\hat{\Pi}_n^{odd}$ and $\hat{\Pi}_n^{even}$ are $(q_{1,n+1}, q_{2,n+1}, \ldots, q_{n,n+1})$ and $(q_{2,n+1}, q_{3,n+1}, \ldots, q_{n+1,n+1})$. Therefore, the two inequalities in (33) cannot be both equality unless there exist some optimal offers $q_{1,n+1}, \ldots, q_{n+1,n+1}$ for the $(n+1)$-variety problem such that

1. $q_{1,n+1}, q_{2,n+1}, \ldots, q_{n,n+1}$ are some optimal offers for the n-variety problem;

2. $q_{2,n+1}, q_{3,n+1}, \ldots, q_{n+1,n+1}$ are some optimal offers for the n-variety problem; and

3. $q_{2,n+1}, q_{3,n+1}, \ldots, q_{n-1,n+1}$ are some optimal offers for the $(n-1)$-variety problem.

However, checking our first-order conditions/difference equations (13) and (14), these are impossible. ■

Proof of Theorem 2. From (27),

$$\Pi_\infty = \int_{x_*}^{\bar{t}} \max_{q \geq 0} H\left(q, x\right) dG(x) = \int_{x_*}^{\bar{t}} H\left(\tilde{q}(x), x\right) dG(x),$$

where $\tilde{q}(x) \equiv q_\infty(J^{-1}(x))$ for every $x \in [x_*, \bar{t}]$, or equivalently

$$xv'(\tilde{q}(x)) = c'(\tilde{q}(x)).$$

Given a number of varieties n, any discrete offering can be expressed in terms virtual types instead of types. Let q_i be the quality to serve consumers with virtual types $x \in [x_i, x_{i+1}]$. Now consider the (suboptimal) discrete offering characterized by

$$x_i = \frac{c'(0)}{v'(0)} + \frac{2i-1}{2n+1}\left(\bar{t} - \frac{c'(0)}{v'(0)}\right),$$

$$q_i = \tilde{q}\left(\frac{x_i + x_{i+1}}{2}\right).$$

Then the corresponding (suboptimal) n-variety profit is

$$\hat{\Pi}_n = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} H(q_i, x) dG(x).$$

Subtracting $\hat{\Pi}_n$ from Π_∞ and noticing that $x_* = c'(0)/v'(0) < x_1 < \cdots < x_n < x_{n+1} = \bar{t}$, we have

$$\Pi_\infty - \hat{\Pi}_n = \int_{x_*}^{x_1} H(\tilde{q}(x), x) dG(x) + \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left[H(\tilde{q}(x), x) - H(q_i, x)\right] dG(x). \tag{34}$$

The integrand of the first term of the right-hand side of (34), as a twice continuously differentiable function of x, can be Taylor expanded around x_* as

$$x_* v(\tilde{q}(x_*)) - c(\tilde{q}(x_*)) + \left(x - x_*\right) v(\tilde{q}(x_*)) + \frac{1}{2}\left(x - x_*\right)^2 v'(\tilde{q}(\hat{x}_0))\tilde{q}'(\hat{x}_0)$$

for some $\hat{x}_0 \in [x_*, x_1]$. From the fact that $\tilde{q}(x_*) = \tilde{q}(c'(0)/v'(0)) = 0$, it is further simplified as

$$\frac{1}{2}\left(x - \frac{c'(0)}{v'(0)}\right)^2 v'(\tilde{q}(\hat{x}_0))\tilde{q}'(\hat{x}_0).$$

The integrand of other terms of the right-hand side of (34), as a twice continuously differentiable function of x, can be Taylor expanded around $(x_i + x_{i+1})/2$ as

$$\begin{aligned}
& \left(\frac{x_i + x_{i+1}}{2} v(q_i) - c(q_i)\right) - \left(\frac{x_i + x_{i+1}}{2} v(q_i) - c(q_i)\right) \\
& + \left(x - \frac{x_i + x_{i+1}}{2}\right)(v(q_i) - v(q_i)) + \frac{1}{2}\left(x - \frac{x_i + x_{i+1}}{2}\right)^2 v'(\tilde{q}(\hat{x}_i))\tilde{q}'(\hat{x}_i) \\
= \; & \frac{1}{2}\left(x - \frac{x_i + x_{i+1}}{2}\right)^2 v'(\tilde{q}(\hat{x}_i))\tilde{q}'(\hat{x}_i)
\end{aligned}$$

for some $\hat{x}_i \in [x_i, x_{i+1}]$. By definition of $\tilde{q}(\cdot)$, for any $x \in [x_*, \bar{t}]$,

$$v'(\tilde{q}(x)) + xv''(\tilde{q}(x))\tilde{q}'(x) = c''(\tilde{q}(x))\tilde{q}'(x)$$

and hence

$$\begin{aligned}
v'(\tilde{q}(x))\tilde{q}'(x) &= \frac{(v'(\tilde{q}(x)))^2}{c''(\tilde{q}(x)) - xv''(\tilde{q}(x))} \\
&\leq \sup_{x \in [x_*, \bar{t}]} \left\{ \frac{(v'(\tilde{q}(x)))^2}{c''(\tilde{q}(x)) - xv''(\tilde{q}(x))} \right\} \\
&= \sup_{t \in [t_*, \bar{t}]} \left\{ \frac{(v'(q_\infty(t)))^2}{c''(q_\infty(t)) - J(t)v''(q_\infty(t))} \right\}.
\end{aligned}$$

Since $J(t) = c'(q_\infty(t))/v'(q_\infty(t))$,

$$\begin{aligned}
v'(\tilde{q}(x))\tilde{q}'(x) &\leq \sup_{t \in [t_*, \bar{t}]} \left\{ \frac{(v'(q_\infty(t)))^3}{v'(q_\infty(t))c''(q_\infty(t)) - c'(q_\infty(t))v''(q_\infty(t))} \right\} \\
&= \sup_{q \in [0, q(\bar{t})]} \left\{ \frac{(v'(q))^3}{v'(q)c''(q) - c'(q)v''(q)} \right\} \equiv M_1.
\end{aligned}$$

Notice that $M_1 < \infty$ because it is the supremum of a continuous function over a compact set.

Now, the first term of the right-hand side of (34) is

$$\begin{aligned} & \int_{c'(0)/v'(0)}^{x_1} \frac{1}{2}\left(x - \frac{c'(0)}{v'(0)}\right)^2 v'(\tilde{q}(\hat{x}_0))\tilde{q}'(\hat{x}_0)dG(x) \\ \leq & \int_{c'(0)/v'(0)}^{x_1} \frac{1}{2}\left(x - \frac{c'(0)}{v'(0)}\right)^2 M_1 dG(x) \\ < & \int_{c'(0)/v'(0)}^{x_1} \frac{1}{2}\left(x_1 - \frac{c'(0)}{v'(0)}\right)^2 M_1 dG(x) \\ = & \frac{M_1}{2(2n+1)^2}\left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^2 \left(G\left(x_1\right) - G(x_*)\right). \end{aligned} \tag{35}$$

Other terms of the right-hand side of (34) are

$$\begin{aligned} & \int_{x_i}^{x_{i+1}} \frac{1}{2}\left(x - \frac{x_i + x_{i+1}}{2}\right)^2 v'(\tilde{q}(\hat{x}_i))\tilde{q}'(\hat{x}_i)dG(x) \\ \leq & \int_{x_i}^{x_{i+1}} \frac{1}{2}\left(x - \frac{x_i + x_{i+1}}{2}\right)^2 M_1 dG(x) \\ < & \int_{x_i}^{x_{i+1}} \frac{1}{2}\max\left\{\left(x_i - \frac{x_i + x_{i+1}}{2}\right)^2, \left(x_{i+1} - \frac{x_i + x_{i+1}}{2}\right)^2\right\} M_1 dG(x) \\ = & \frac{M_1}{2(2n+1)^2}\left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^2 \left(G\left(x_{i+1}\right) - G(x_i)\right). \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} \Pi_\infty - \Pi_n & \leq \Pi_\infty - \hat{\Pi}_n \\ & < \frac{M_1}{2(2n+1)^2}\left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^2 \left(G\left(x_1\right) - G(x_*) + \sum_{i=1}^{n}\left(G\left(x_{i+1}\right) - G(x_i)\right)\right) \\ & = \frac{M_1}{2(2n+1)^2}\left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^2 \left(G\left(\bar{t}\right) - G(x_*)\right) \\ & = \frac{M_1}{2(2n+1)^2}\left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^2 \left(1 - F(t_*)\right). \end{aligned}$$

It proves (30) with

$$M_0 \equiv \frac{M_1}{8} \cdot \left(\bar{t} - \frac{c'(0)}{v'(0)}\right)^2 \left(1 - F(t_*)\right).$$

Moreover, if $G(\cdot)$ admits a density $g(\cdot)$ over the support $[x_*, \bar{t}]$ that is bounded from above by $\bar{g}$, then line (35) is at most

$$\begin{aligned} \frac{1}{2} M_1 \bar{g} \int_{c'(0)/v'(0)}^{x_1} \left(x - \frac{c'(0)}{v'(0)} \right)^2 dx &= \frac{1}{6} M_1 \bar{g} \left(x_1 - \frac{c'(0)}{v'(0)} \right)^3 \\ &= \frac{M_1 \bar{g}}{6 (2n+1)^3} \left(\bar{t} - \frac{c'(0)}{v'(0)} \right)^3, \end{aligned}$$

and line (36) is at most

$$\begin{aligned} \frac{1}{2} M_1 \bar{g} \int_{x_i}^{x_{i+1}} \left(x - \frac{x_i + x_{i+1}}{2} \right)^2 dx &= \frac{1}{6} M_1 \bar{g} \left[\left(\frac{x_{i+1} - x_i}{2} \right)^3 - \left(\frac{x_i - x_{i+1}}{2} \right)^3 \right] \\ &= \frac{2 M_1 \bar{g}}{6 (2n+1)^3} \left(\bar{t} - \frac{c'(0)}{v'(0)} \right)^3. \end{aligned}$$

Therefore,

$$\begin{aligned} \Pi_\infty - \Pi_n &\leq \Pi_\infty - \hat{\Pi}_n \\ &\leq \frac{(2n+1) M_1 \bar{g}}{6 (2n+1)^3} \left(\bar{t} - \frac{c'(0)}{v'(0)} \right)^3 \\ &= \frac{M_1 \bar{g}}{6 (2n+1)^2} \left(\bar{t} - \frac{c'(0)}{v'(0)} \right)^3. \end{aligned}$$

■

Proof of Lemma 1. The maximized values Π_∞, $\hat{\Pi}_1$ and $\hat{\Pi}_2$ can be solved as

$$\Pi_\infty = \frac{1}{2A_2} \int_{t_*}^{\bar{t}} (J(t) A_0 - A_1)^2 \, dF(t) = \frac{1}{2A_2} \int_{x_*}^{\bar{t}} (x A_0 - A_1)^2 \, dG(x),$$

$$\hat{\Pi}_1 = \frac{1}{2A_2} \frac{\left[\int_{x_*}^{\bar{t}} (x A_0 - A_1) \, dG(x) \right]^2}{1 - G(x_*)},$$

$$\hat{\Pi}_2 = \frac{1}{2A_2} \max_{\hat{x} \in [x_*, \bar{t}]} \left\{ \frac{\left[\int_{x_*}^{\hat{x}} (x A_0 - A_1) \, dG(x) \right]^2}{G(\hat{x}) - G(x_*)} + \frac{\left[\int_{\hat{x}}^{\bar{t}} (x A_0 - A_1) \, dG(x) \right]^2}{1 - G(\hat{x})} \right\}.$$

For any $\hat{x} \in [x_*, \bar{t}]$,

$$\begin{aligned} 2A_2\Pi_\infty &= \int_{x_*}^{\hat{x}} (xA_0 - A_1)^2 \, dG(x) + \int_{\hat{x}}^{\bar{t}} (xA_0 - A_1)^2 \, dG(x) \\ &= (\hat{x}A_0 - A_1)^2 (G(\hat{x}) - G(x_*)) - 2A_0 \int_{x_*}^{\hat{x}} \frac{G(x) - G(x_*)}{g(x)} (xA_0 - A_1) \, dG(x) \\ &\quad + (\hat{x}A_0 - A_1)^2 (1 - G(\hat{x})) + 2A_0 \int_{\hat{x}}^{\bar{t}} \frac{1 - G(x)}{g(x)} (xA_0 - A_1) \, dG(x) \\ &= (\hat{x}A_0 - A_1)^2 (1 - G(x_*)) \\ &\quad -2A_0 (G(\hat{x}) - G(x_*)) \int_{x_*}^{\hat{x}} \frac{G(x) - G(x_*)}{g(x)} (xA_0 - A_1) \frac{dG(x)}{G(\hat{x}) - G(x_*)} \\ &\quad +2A_0 (1 - G(\hat{x})) \int_{\hat{x}}^{\bar{t}} \frac{1 - G(x)}{g(x)} (xA_0 - A_1) \frac{dG(x)}{1 - G(\hat{x})}. \end{aligned} \tag{37}$$

Since both $(G(x) - G(x_*))/g(x)$ and $xA_0 - A_1$ are nondecreasing in x, they have nonnegative covariance when x is randomly drawn from $[x_*, \hat{x}]$. Thus, the integral in the second line of (37) is at least

$$\int_{x_*}^{\hat{x}} \frac{G(x) - G(x_*)}{g(x)} \frac{dG(x)}{G(\hat{x}) - G(x_*)} \cdot \int_{x_*}^{\hat{x}} (xA_0 - A_1) \frac{dG(x)}{G(\hat{x}) - G(x_*)}.$$

Since $(1 - G(x))/g(x)$ is nonincreasing and $xA_0 - A_1$ is nondecreasing in x, they have nonpositive covariance when x is randomly drawn from $[\hat{x}, \bar{t}]$. Thus, the integral in the third line of (37) is at most

$$\int_{\hat{x}}^{\bar{t}} \frac{1 - G(x)}{g(x)} \frac{dG(x)}{1 - G(\hat{x})} \cdot \int_{\hat{x}}^{\bar{t}} (xA_0 - A_1) \frac{dG(x)}{1 - G(\hat{x})}.$$

Therefore,

$$\begin{aligned} 2A_2\Pi_\infty &\leq (\hat{x}A_0 - A_1)^2 (1 - G(x_*)) \\ &\quad - \frac{2A_0}{G(\hat{x}) - G(x_*)} \int_{x_*}^{\hat{x}} \frac{G(x) - G(x_*)}{g(x)} dG(x) \cdot \int_{x_*}^{\hat{x}} (xA_0 - A_1) \, dG(x) \\ &\quad + \frac{2A_0}{1 - G(\hat{x})} \int_{\hat{x}}^{\bar{t}} \frac{1 - G(x)}{g(x)} dG(x) \cdot \int_{\hat{x}}^{\bar{t}} (xA_0 - A_1) \, dG(x). \end{aligned} \tag{38}$$

The first integral in the second line of (38) can be written as

$$\hat{x} (G(\hat{x}) - G(x_*)) - \int_{x_*}^{\hat{x}} x dG(x),$$

and the first integral in the third line of (38) can be written as

$$-\hat{x}\left(1-G(\hat{x})\right)+\int_{\hat{x}}^{\bar{t}} x dG(x).$$

Therefore,

$$\begin{aligned}
2A_2\Pi_\infty \leq\ & (\hat{x}A_0-A_1)^2\left(1-G(x_*)\right) \\
& -2A_0\left(\hat{x}-\frac{\int_{x_*}^{\hat{x}} x dG(x)}{G(\hat{x})-G(x_*)}\right)\cdot\int_{x_*}^{\hat{x}}(xA_0-A_1)\,dG(x) \\
& -2A_0\left(\hat{x}-\frac{\int_{\hat{x}}^{\bar{t}} x dG(x)}{1-G(\hat{x})}\right)\cdot\int_{\hat{x}}^{\bar{t}}(xA_0-A_1)\,dG(x). \qquad (39)
\end{aligned}$$

The second line of (39) can be written as

$$-2\left(\hat{x}A_0-A_1\right)\int_{x_*}^{\hat{x}}(xA_0-A_1)\,dG(x)+\frac{2\left[\int_{x_*}^{\hat{x}}(xA_0-A_1)\,dG(x)\right]^2}{G(\hat{x})-G(x_*)}$$

and the third line of (39) can be written as

$$-2\left(\hat{x}A_0-A_1\right)\int_{\hat{x}}^{\bar{t}}(xA_0-A_1)\,dG(x)+\frac{2\left[\int_{\hat{x}}^{\bar{t}}(xA_0-A_1)\,dG(x)\right]^2}{1-G(\hat{x})},$$

and

$$\frac{\left[\int_{x_*}^{\hat{x}}(xA_0-A_1)\,dG(x)\right]^2}{G(\hat{x})-G(x_*)}+\frac{\left[\int_{\hat{x}}^{\bar{t}}(xA_0-A_1)\,dG(x)\right]^2}{1-G(\hat{x})}\leq 2A_2\hat{\Pi}_2.$$

Therefore,

$$2A_2\Pi_\infty \leq (\hat{x}A_0-A_1)^2\left(1-G(x_*)\right)-2\left(\hat{x}A_0-A_1\right)\int_{x_*}^{\bar{t}}(xA_0-A_1)\,dG(x)+4A_2\hat{\Pi}_2.$$

Since the above holds for any $\hat{x}\in[x_*,\bar{t}]$, let us from now on take the unique $\hat{x}$ that solves

$$\hat{x}A_0-A_1=\frac{\int_{x_*}^{\bar{t}}(xA_0-A_1)\,dG(x)}{1-G(x_*)}.$$

Then,

$$2A_2\Pi_\infty \leq -\frac{\left[\int_{x_*}^{\bar{t}}(xA_0-A_1)\,dG(x)\right]^2}{1-G(x_*)}+4A_2\hat{\Pi}_2=-2A_2\hat{\Pi}_1+4A_2\hat{\Pi}_2.$$

■

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