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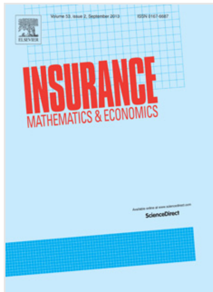

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Confidence Band for Expectation Dependence with Applications

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Abstract

Motivated by the applications of the concept of expectation dependence in economics and finance, we propose a method to construct uniform confidence band for expectation dependence. It is derived based on Hoeffding's inequality. Our proposed confidence band can be explicitly expressed and thus it is very easy to implement. Our method has applications to demand for a risky asset and first-order risk aversion problems. Simulations suggest our proposed confidence interval can control the coverage probabilities very well, and the average lengths are very short. Two empirical applications are presented to illustrate the usefulness of the constructed confidence band of expectation dependence.

Key words: Expectation dependence; Confidence band estimation; Demand for a risky asset; First-order risk aversion.

JEL classification: C14, C13, D81

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1 Introduction

Since Galton (1886) coined the concept of correlation, it serves as a popular measure of dependence in many economic and financial studies. However, for non-normal distributions, correlation is often too weak to imply meaningful conclusions. In order to study on stronger definitions of dependence, Lehmann (1966) makes far-reaching contributions to the characterization of quadrant dependence. For literatures about this concept, see for instance, Denuit and Scaillet (2004), Scaillet (2005), Kallenberg (2008), Dhaene et al. (2009), Gijbels et al. (2010) and Ledwina and Wylupek (2014).

In some situations, a less restrictive measure of dependence than quadrant dependence can be useful to obtain explicit results. Wright (1987) proposes the concept of expectation dependence (ED) which is a weaker definition of dependence. For two random variables X and Y, he interprets negative ED as follows: "When we discover Y is small, in the precise sense that we are given the truncation $Y \leq y$, our expectation of X is revised upward." Though ED is a weaker definition of dependence than quadrant dependence, it is a stronger definition than correlation. He also shows ED is a key in portfolio theory.

However, the literature didn't pay much attention to ED until Hong et al. (2011) and Li (2011) bring back to life the concept of ED. Hong et al. (2011) show an individual will purchase less than full (more than full) insurance if and only if the insurable risk is positively (negatively) expectation dependent with random initial wealth. Li (2001) shows ED is at the core of condition for aversion (liking) of a background risk. Since then ED has been used in many economic and financial studies. For example, Wong (2013) shows ED plays a pivotal role in determining the bank's optimal choice between fixed and variable rate loans; Wong (2012 a, b, c; 2014 a, b) finds expectation dependence are useful in determining the firm's optimal hedging position; Using the concept of ED, Dionne and Li (2014) show first-order conditional dependent risk aversion is consistent with the framework of the expected utility hypothesis.

Note that a number of problems in economics, finance, insurance, and generally in decision making under uncertainty rely on estimates of the covariance between (transformed) random variables, which can, for example, be losses, risks, incomes, financial returns, and so forth. Egozcue et al. (2011) sharpen the upper bound of the covariance between (transformed) random variables by incorporating the notion of ED. Besides, Egozcue et al. (2013) further establish general results that determine when convex combinations of arbitrary quadrant dependence copulas give rise to ED copulas.

Several studies also weaken ED. For example, Li (2011) proposes the concept of higher-order ED; Denuit et al. (2015) develop almost ED concept. They also give some interesting economic interpretations and applications for these concepts.

Recently, Zhu et al. (2015) propose some consistent test statistics for ED. Compared with test statistic, confidence bands can generally tell us more information. Test statistics can only show whether there is ED. While, confidence bands can inform more about the extent of ED. The confidence band can not only be used to test whether there is ED, but also can tell us where ED is violated. Furthermore, in many applications, see for instance Section 4, we are not directly interested in the EDs. Instead, we would like to make inference on some functionals of the EDs. Whether these functionals are positive or not is not equivalent to whether there are EDs. Thus, Zhu et al. (2015)'s test statistics can not be directly used since they only focus on testing ED. Instead, confidence intervals for these functionals are required. In this paper, we aim to construct confidence bands for ED. Thereafter, we can construct confidence intervals for functionals of ED. Instead of pointwise confidence interval, uniform confidence band is investigated. This makes the construction difficult. To this end, we apply Hoeffding's inequality. The constructed confidence band has simple form and is easy to implement.

We use two applications to show how to apply our result to economic and financial studies. We first construct semiparametric confidence bands for the demand for a risky asset problem. Then we construct semiparametric confidence bands for first-order conditional dependent risk aversion. These applications show how to combine our confidence bands with economics theories to obtain the precise answers of economic and finance problems.

The paper proceeds as follows. Section 2 reviews the concept of ED. Section 3 constructs confidence bands for ED. Section 4 discusses two applications. Section 5 presents a simulation study. Section 6 conducts empirical studies. Section 7 concludes this paper.

2 The concept of expectation dependence

Suppose $X \times Y \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$ be a 2-dimensional random vector. Wright (1987) proposes the following concept.

Definition 2.1 *(Wright 1987) If*

$$ED(X|Y \leq y) = E(X) - E(X|Y \leq y) \geq 0 \ for \ all \ y, \tag{1}$$

then X is positive expectation dependent on Y. Negative expectation dependence is defined analogously if we reverse the sign of the inequality in (1).

Wright (1987) interprets $ED(X|Y \leq y) \geq 0$ as: when we know that Y is truncated from above ($Y \leq y$), the expectation of X decreases.

Interestingly, ED can be restated in terms of covariance, as shown in Denuit et al. (2015). Positive ED can be rewritten as, for all y,

$$ED(X|y) \geq 0 \Leftrightarrow -cov(X, I(y - Y \geq 0)) \geq 0, \tag{2}$$

where $I(\mathcal{E})$ is the indicator function that is equal to 1 if the event $\mathcal{E}$ occurs and 0 otherwise. We see from (2) that positive ED is equivalent to minus the covariance between X and the payoff of a digital option protecting against a shortfall of Y below y.

Another notable feature of ED is that ED ensures that $cov(X, t(Y)) \geq 0$ for all non-decreasing transformations $t(\cdot)$ of Y. This is established formally by Wright (1987).

3 Construction of confidence band

Let (X_i, Y_i) $i = 1, .., n$ be an i.i.d sample from (X, Y). Note that

$$\begin{aligned} ED(X|Y \leq y) &= E(X) - E(X|Y \leq y) \\ &= E(X) - \frac{E(XI(Y \leq y))}{E(I(Y \leq y))} \\ &= -\frac{E(\{X - E(X)\}I(Y \leq y))}{E(I(Y \leq y))}. \end{aligned}$$

Thus $ED(y) := ED(X|Y \leq y)$ or $PED(y) := ED(X|Y \leq y) \times E(I(Y \leq y))$ can be estimated easily by using sample average. To be precise, we can have:

$$\begin{aligned} \widehat{ED}(y) &= \frac{n^{-1}\sum_{i=1}^n \{X_i - \bar{X}\}I(Y_i \geq y)}{n^{-1}\sum_{i=1}^n I(Y_i \leq y)}; \\ \widehat{PED}(y) &= n^{-1}\sum_{i=1}^n \{X_i - \bar{X}\}I(Y_i \geq y). \end{aligned}$$

Here $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. Note that $PED(y)$ is more convenient to use in practice. Also for many applications, see for instance Section 4, using $ED(y)$ or $PED(y)$ is equivalent. Thus in the following, we focus on the construction of confidence band for $PED(y)$.

First note that

$$\begin{aligned}
\widehat{PED}(y) &= \frac{1}{n}\sum_{i=1}^{n}\{X_i-\bar{X}\}I(Y_i \geq y) \\
&= \frac{1}{n}\sum_{i=1}^{n}\{X_i-E(X)\}I(Y_i \geq y)+\frac{1}{n}\sum_{i=1}^{n}\{E(X)-\bar{X}\}I(Y_i \geq y) \\
&= \frac{1}{n}\sum_{i=1}^{n}\{X_i-E(X)\}\{I(Y_i \geq y)-E(I(Y_i \geq y))\} \\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\{E(X)-\bar{X}\}\{I(Y_i \geq y)-E(I(Y_i \geq y))\} \\
&:= \frac{1}{n}\sum_{i=1}^{n} l_i(y)+O_p(\frac{1}{n}).
\end{aligned}$$

Here $l_i(y)=\{X_i-E(X)\}\{I(Y_i \geq y)-E(I(Y_i \geq y))\}$. The last equation holds uniformly.

We first propose the following result

Lemma 3.1

$$Pr\left(\sup_y |\frac{1}{n}\sum_{i=1}^{n} l_i(y)-PED(y)| > \epsilon\right) \leq 2\exp\left(-\frac{2n\epsilon^2}{C^2}\right), \tag{3}$$

where $C=2\max\{E(X)-\underline{x},\bar{x}-E(X)\}$.

Proof See appendix. Q.E.D.

When X follows a symmetry distribution, $E(X)-\underline{x}=\bar{x}-E(X)$ and thus $C=\bar{x}-\underline{x}$. For $X \sim N(\mu,\sigma^2)$, the support of X is unbounded. However, note that, $P(|X-\mu| \leq 3\sigma)=0.9973$. Thus X can be approximatively considered to be bounded. And C can be taken to be 6σ. When C involves unknown parameters, such as, $E(X)$ and σ, we can estimate C by plugging in the corresponding estimators of the parameters.

Let $2\exp\left(-\frac{2n\epsilon_0^2}{C^2}\right)=\alpha$, we can get $\epsilon_0=C\sqrt{\frac{1}{2n}\log\frac{2}{\alpha}}$ and finally the confidence band for $PED(y)$ based on $n^{-1}\sum_{i=1}^{n} l_i(y)$ easily. This result is not based on asymptotic theory but is for finite sample. However, $l_i(y)$ is unknown and has to be estimated in practice.

Now we can have that:

$$\begin{aligned}
& Pr\left(\sup_y |\widehat{PED}(y)-PED(y)| > \epsilon\right) \\
\leq\ & Pr\left(\sup_y |\frac{1}{n}\sum_{i=1}^{n} l_i(y)-PED(y)| > \epsilon-\frac{c_n}{n}\right) \\
\leq\ & 2\exp\left(-\frac{2n(\epsilon-c_n/n)^2}{C^2}\right).
\end{aligned}$$

Here $c_n \geq 0$ and are bounded.

Let $\epsilon = \epsilon_0 = C\sqrt{\frac{1}{2n}\log\frac{2}{\alpha}}$, $c_n/n = O(1/n) = o(\epsilon_0)$. Thus for relatively large sample size, c_n/n can be ignored and approximately we can also have:

$$Pr\left(\sup_y |\widehat{PED}(y) - PED(y)| > \epsilon_0\right) \leq 2\exp\left(-\frac{2n(\epsilon_0 - c_n/n)^2}{C^2}\right) \approx \alpha.$$

Therefore, we obtain:

Proposition 3.2 *Let $L(y) = \widehat{PED}(y) - \epsilon_0$ and $U(y) = \widehat{PED}(y) + \epsilon_0$, then*

$$Pr\Big(\forall\, y, L(y) \leq PED(y) \leq U(y)\Big) \geq 1 - 2\exp\left(-\frac{2n(\epsilon_0 - c_n/n)^2}{C^2}\right) \approx 1 - \alpha.$$

Remark If we are only interested in pointwise confidence interval for $PED(y)$, we can use asymptotic distribution for $\widehat{PED}(y)$ at any fixed point y. This approach is relatively easier. However, for applications in next Section, uniform confidence band is needed. One-side confidence band can be similarly defined. The details are omitted here.

4 Applications

In this section, we illustrate the applicability of our result to two problems. In particular, we demonstrate how to construct semiparametric confidence bands. In these applications, the risks are distribution free while the utility functions are known.

4.1 The demand for a risky asset in the presence of a background risk

The theory of the demand for a risky asset is a key of diversification. It is applicable in portfolio choices, production decisions and insurance decisions. For these applications, we need to know precisely if we should take a risk.

We consider an agent with a bivariate expected utility $u(w, y)$. Let u_1 denote $\frac{\partial u(w,y)}{\partial w}$, u_2 denote $\frac{\partial u(w,y)}{\partial y}$, u_{11} denote$\frac{\partial^2 u(w,y)}{\partial w^2}$ and u_{12} denote$\frac{\partial^2 u(w,y)}{\partial w \partial y}$. W represents wealth and Y is a risk. All distributions are assumed bounded on some finite support, and utility and its derivatives are assume bounded as well. We also assume $u_1 \geq 0$ and $u_{11} \leq 0$.

This agent has to allocate a sure wealth w between a safe asset paying a return r_f and a risky one paying a random return R. She wants to choose θ, which measures the extent of risk taking, to maximize expected utility. The problem can be written in the following compact manner

$$V(\theta) = E(u(W, Y)) = E(u(w_0 + \theta X, Y)), \tag{4}$$

where $w_0 = w(1 + r_f)$ and $X = R - r_f$. Without loss of generality, we also assume $E(X) \geq 0$.

Define θ^* as the solution to this problem. We wish to find conditions under which $\theta^* \geq 0$ which means some risky assets will be purchased. We first recall the following results:

Proposition 4.1 *(Li 2011)*

$\theta^* \geq 0$, if and only if $V'(0) \geq 0$, where

$$V'(0) = E(u_1(w_0, Y))E(X) + \int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0, Y)F_Y(y)dy. \tag{5}$$

$V'(0)$ is the marginal expected utility for purchasing the first unit of a risky asset. Hence Proposition 4.1 states, if the marginal expected utility for purchasing the first unit of a risky asset is positive, then the agent should purchase it.

Suppose $u_{12} \geq 0$ for all $x \times y \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$, then we can use Zhu et al. (2015)'s consistent test to test:

$$H_0 : ED(y) \geq 0 \quad for\ all\ y. \tag{6}$$

If we cannot reject H_0, then we cannot reject the hypothesis: $V'(0) \geq 0$. Therefore the agent should purchase the risky asset.

However, we should note that whether $V'(0) \geq 0$ is not equivalent to whether $ED(y) \geq 0$. If the above H_0 is rejected, it does not imply $V'(0)$ is not positive. The integral

$$\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0, Y)F_Y(y)dy$$

can be positive even there are some $ED(y) < 0$. Consider the almost ED concept introduced by Denuit et al. (2015). Let $\Omega = \{y : ED(y) < 0\}$. The almost ED concept asks that

$$-\int_{\Omega} ED(y)F_Y(y)dy \leq \kappa \left(-\int_{\Omega} ED(y)F_Y(y)dy + \int_{\Omega^c} ED(y)F_Y(y)dy \right).$$

From this, we can easily obtain that $\int_{\Omega^c} ED(y)F_Y(y)dy \geq -(1/\kappa - 1) \int_{\Omega} ED(y)F_Y(y)dy$. And thus $\int_{\underline{y}}^{\bar{y}} ED(y)F_Y(y)dy \geq -(1/\kappa - 2) \int_{\Omega} ED(y)F_Y(y)dy \geq 0$ as long as $\kappa < 1/2$. Thus to ensure the integral to be positive, we do not need ED holds for all y. This explanation also applies to the integral $\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0, Y)F_Y(y)dy$.

Furthermore even if this integral is negative, the $V'(0)$, as a sum of a positive value $E(u_1(w_0, Y))E(X)$ and the integral $\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0, Y)F_Y(y)dy$, can still be positive as long as the absolute value of the former is larger.

On the other hand, there may be some agents such that $u_{12} < 0$ for some (x, y). In sum, we can not assert $V'(0) \geq 0$ through testing H_0. Thus a confidence interval is needed.

From Proposition 3.2, we can obtain the following confidence interval for $V'(0)$.

Proposition 4.2

$$Pr\left(J \leq V'(0) \leq I\right) \geq 1 - \alpha, \quad as\ n \to \infty; \tag{7}$$

where

$$J = \frac{1}{n}\sum_{i=1}^{n} X_i u_1(w_0, Y_i) + \epsilon_0[u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \tag{8}$$

and

$$I = \frac{1}{n}\sum_{i=1}^{n} X_i u_1(w_0, Y_i) - \epsilon_0[u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})]. \tag{9}$$

Proof See appendix. Q.E.D.

From above proposition, we know, if $J \geq 0$, then $V'(0) \geq 0$ (the agent should purchase the risky asset) with confidence $1 - \alpha$.

If further assumption is made on the form of utility function, we can construct semiparametric confidence bands via Proposition 4.2.

We use the following two particular types of utility functions that are often encountered in the economics and the finance literature to show how to apply Proposition 4.2.

• Constant absolute risk aversion (CARA) utility function: Let $u(w, y) = 1 - \exp^{-\lambda(w+y)}$. Then $u_1(w, y) = \lambda \exp^{-\lambda(w+y)}$. Hence

$$J = \frac{\lambda}{n}\sum_{i=1}^{n} X_i \exp^{-\lambda(w_0+Y_i)} + \lambda\epsilon_0[\exp^{-\lambda(w_0+\bar{y})} - \exp^{-\lambda(w_0+\underline{y})}];$$

$$I = \frac{\lambda}{n}\sum_{i=1}^{n} X_i \exp^{-\lambda(w_0+Y_i)} - \lambda\epsilon_0[\exp^{-\lambda(w_0+\bar{y})} - \exp^{-\lambda(w_0+\underline{y})}].$$

• Constant relative risk aversion (CRRA) utility function: When $u(w, y) = \frac{(w+y)^{1-\gamma}}{1-\gamma}$. Then $u_1(w, y) = (w + y)^{-\gamma}$. Thus

$$J = \frac{1}{n}\sum_{i=1}^{n} X_i(w_0 + Y_i)^{-\gamma} + \epsilon_0[(w_0 + \bar{y})^{-\gamma} - (w_0 + \underline{y})^{-\gamma}];$$

$$I = \frac{1}{n}\sum_{i=1}^{n} X_i(w_0 + Y_i)^{-\gamma} - \epsilon_0[(w_0 + \bar{y})^{-\gamma} - (w_0 + \underline{y})^{-\gamma}].$$

4.2 First-order risk aversion

First-order risk aversion means small risks matter. It has been used to explain puzzles in the economic and financial literatures. In many situations, we need to know precisely if an agent is first-order risk averse.

Define $Z = m\varepsilon$ as the risk faced by an agent. The size of the risk is measured by parameter m. One way to measure the agent's degree of risk aversion for Z is to ask her how much she is willing to pay to eliminate Z. This value is defined as the risk premium $\pi(m)$ associated with that risk. For an agent with utility function u, the risk premium $\pi(m)$ is defined by the following equation:

$$u(w + E(m\varepsilon) - \pi(m), E(Y)) = E\Big(u(w + m\varepsilon, E(Y))\Big). \tag{10}$$

where $E(Y)$ is the expected value of another risk Y and w is non-random initial wealth.

Definition 4.3 *(Segal and Spivak, 1990) The agent's attitude towards risk at w is first-order if for every ε with $E(\varepsilon) = 0$, $\pi'(0) \neq 0$. The agent's attitude towards risk at w is second-order if for every ε with $E(\varepsilon) = 0$, $\pi'(0) = 0$ but $\pi''(0) \neq 0$.*

First-order risk aversion means small risks matter.

By considering the characteristics of $\pi(m)$ in the presence of an independent uninsured risk, Loomes and Segal (1994) propose the order of conditional risk aversion. For an agent, the conditional risk premium $\pi_c(m)$ is defined by the following equation:

$$E\Big(u(w + E(m\varepsilon) - \pi_c(m), Y^i)\Big) = E(u(w + m\varepsilon, Y^i)), \tag{11}$$

where Y^i is an independent uninsured risk.

Definition 4.4 *(Loomes and Segal, 1994) The agent's attitude towards risk at w is first-order conditional risk aversion if for every ε with $E(\varepsilon) = 0$, $\pi_c'(0) \neq 0$. The agent's attitude towards risk at w is second-order conditional risk aversion if for every ε with $E(\varepsilon) = 0$, $\pi_c'(0) = 0$ but $\pi_c''(0) \neq 0$.*

Dionne and Li (2014) define conditional dependent risk premium, $\pi_{cd}(m)$ by the following equation:

$$E\Big(u(w + E(m\varepsilon) - \pi_{cd}(m), Y)\Big) = E(u(w + m\varepsilon, Y)), \tag{12}$$

when Y can be a dependent uninsured risk, and propose the following definitions:

Definition 4.5 *(Dionne and Li 2014) The agent's attitude towards risk at w is first-order conditional dependent risk aversion if for every ε, $\pi_{cd}(m) - \pi_c(m) = O(m)$. The agent's attitude towards risk at w is second-order conditional dependent risk aversion if for every ε, $\pi_{cd}(m) - \pi_c(m) = O(m^2)$.*

They also obtain the following result.

Lemma 4.6

$$\pi_{cd}(m) = -m\frac{\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w,Y)F_Y(y)dy}{E(u_1(w,Y))} + O(m^2), \tag{13}$$

Suppose $u_{12} \leq (\geq)0$ for all $x \times y \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$, then we can use Zhu et al. (2015)'s consistent test to test:

$$H_0 : ED(y) \geq 0 \quad \text{for all } y. \tag{14}$$

If we cannot reject H_0, then we cannot reject the hypothesis: $\pi_{cd}(m) \geq (\leq)0$. Therefore we cannot say if the agent is positive (negative) first-order conditional dependent risk averse.

However, as we mentioned before, the integral $\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0,Y)F_Y(y)dy$ can be positive even there are some $ED(y) < 0$. In other words, to determine the sign of $\pi_{cd}(m)$, it is not equivalent to verify the sign of $ED(y)$.

Furthermore, there may be some decision makers (DMs) such that $u_{12} > (<)0$ for some (x, y). In sum, we cannot use the above test and a confidence band is useful.

From Proposition 3.2, we can obtain the following confidence interval for $\pi_{cd}(m)$.

Proposition 4.7

$$Pr\left(J \leq \pi_{cd}(m) \leq I\right) \geq 1 - \alpha, \quad \text{as } n \to \infty, \text{ for small } m; \tag{15}$$

where

$$I = (-m)\frac{\frac{1}{n}\sum_{i=1}^{n}\{X_i - \bar{X}\}u_1(w,Y_i) + \epsilon_0[u_1(w,\bar{y}) - u_1(w,\underline{y})]}{\frac{1}{n}\sum_{i=1}^{n}u_1(w,Y_i)} \tag{16}$$

and

$$J = (-m)\frac{\frac{1}{n}\sum_{i=1}^{n}\{X_i - \bar{X}\}u_1(w,Y_i) - \epsilon_0[u_1(w,\bar{y}) - u_1(w,\underline{y})]}{\frac{1}{n}\sum_{i=1}^{n}u_1(w,Y_i)}. \tag{17}$$

The proof in the Proposition 4.7 is similar to that of Proposition 4.2 and is therefore skipped. It is available from the authors upon request.

Proposition 4.7 shows, if $J \geq 0$, then $\pi_{cd}(m) \geq 0$ (the agent is positive first-order conditional dependent risk averse) with confidence $1-\alpha$; if $I \leq 0$, then $\pi_{cd}(m) \leq 0$ (the agent is negative first-order conditional dependent risk averse) with confidence $1-\alpha$.

Two classical utility functions can be considered for Proposition 4.7.

- CARA: $u(w,y) = 1 - \exp^{-\lambda(w+y)}$ and $u_1(w,y) = \lambda \exp^{-\lambda(w+y)}$.

$$I = (-m)\frac{\frac{\lambda}{n}\sum_{i=1}^{n}\{X_i - \bar{X}\}\exp^{-\lambda(w_0+Y_i)} + \lambda\epsilon_0[\exp^{-\lambda(w_0+\bar{y})} - \exp^{-\lambda(w_0+\underline{y})}]}{\frac{\lambda}{n}\sum_{i=1}^{n}\exp^{-\lambda(w_0+Y_i)}};$$

$$J = (-m)\frac{\frac{\lambda}{n}\sum_{i=1}^{n}\{X_i - \bar{X}\}\exp^{-\lambda(w_0+Y_i)} - \lambda\epsilon_0[\exp^{-\lambda(w_0+\bar{y})} - \exp^{-\lambda(w_0+\underline{y})}]}{\frac{\lambda}{n}\sum_{i=1}^{n}\exp^{-\lambda(w_0+Y_i)}}.$$

- CRRA: $u(w,y) = \frac{(w+y)^{1-\gamma}}{1-\gamma}$ and $u_1(w,y) = (w+y)^{-\gamma}$.

$$I = (-m)\frac{\frac{1}{n}\sum_{i=1}^{n}\{X_i - \bar{X}\}(w_0+Y_i)^{-\gamma} + \epsilon_0[(w_0+\bar{y})^{-\gamma} - (w_0+\underline{y})^{-\gamma}]}{\frac{1}{n}\sum_{i=1}^{n}(w_0+Y_i)^{-\gamma}};$$

$$J = (-m)\frac{\frac{1}{n}\sum_{i=1}^{n}\{X_i - \bar{X}\}(w_0+Y_i)^{-\gamma} - \epsilon_0[(w_0+\bar{y})^{-\gamma} - (w_0+\underline{y})^{-\gamma}]}{\frac{1}{n}\sum_{i=1}^{n}(w_0+Y_i)^{-\gamma}}.$$

5 Simulation

Suppose (X,Y) follows the bivariate normal distribution $N_2(\mu,\Sigma)$. Here $\mu = (\mu_1,\mu_2)$ and $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$. Recall $E(X|Y=y) = \mu_1 + \sigma_{12}\sigma_{22}^{-1}(y-\mu_2)$. As a result,

$$E(X|Y \leq y) = \mu_1 + \frac{\sigma_{12}}{\sigma_{22}}\{E(Y|Y \leq y) - \mu_2\}.$$

From Johnson et al. (1994), we know

$$E(Y|Y \leq y) - \mu_2 = -\sigma_{22}\frac{\phi(\beta)}{\Phi(\beta)} \leq 0.$$

Here ϕ and Φ are density function and distribution function of standard normal distribution, respectively; $\beta = (y-\mu_2)/\sigma_{22}$. Thus $ED(y) = \sigma_{12}\frac{\phi(\beta)}{\Phi(\beta)}$. Further note that $\Phi(\beta) = F_Y(y)$ and $\phi(\beta) = \sigma_{22} f_Y(y)$. In other words, $V'(0) = E(u_1(w_0,Y))E(X) + \int_{\underline{y}}^{\bar{y}} u_{12}(w_0,Y)\sigma_{12}\sigma_{22} f_Y(y)dy = E(u_1(w_0,Y))E(X) + \sigma_{12}\sigma_{22}E(u_{12}(w_0,Y))$.

Let $u(w,y) = 1 - \exp^{-\lambda(w+y)}$. Thus $u_1(w_0,Y) = \lambda\exp^{-\lambda(w_0+Y)}$ and $u_{12}(w_0,Y) = -\lambda^2\exp^{-\lambda(w_0+Y)}$. From the moment-generating function of normal distribution, we can easily get $E(\exp^{-\lambda Y}) = \exp^{-\mu_2\lambda + 0.5\sigma_{22}^2\lambda^2}$. Thus

$$V'(0) = \lambda\exp^{-\lambda(w_0+\mu_2)+0.5\sigma_{22}^2\lambda^2}(\mu_1 - \lambda\sigma_{12}\sigma_{22}).$$

Take $\lambda = 0.05, \omega_0 = 0, \mu_i = \sigma_{ii} = 1$. Moreover set $\sigma_{12} = 0, 0.1, \cdots, 0.9$. Sample sizes are taken to be 50 and 100. A total of 2000 Monte Carlo test replications is considered for the 95% two-sided confidence intervals for $V'(0)$.

The simulation results are presented in Table 1. From this table, we can find that first our proposed confidence interval can control the coverage probabilities very well. Second, the average lengths are very short. Moreover, the average lengths does not change with different σ_{12}. With increase of the sample size, the coverage probabilities become closer to the nominal level 0.95. The average lengths also become shorter.

Table 1: The coverage probabilities (CP) and the average lengths (AL) of 95% two-sided confidence intervals for $V'(0)$ with $n = 50$ and 100.

σ_{12}	$CP(n = 50)$	$AL(n = 50)$	$CP(n = 100)$	$AL(n = 100)$
0.0	0.9240	0.0248	0.9620	0.0195
0.1	0.9175	0.0247	0.9595	0.0196
0.2	0.9215	0.0247	0.9565	0.0195
0.3	0.9240	0.0247	0.9545	0.0194
0.4	0.9335	0.0247	0.9545	0.0195
0.5	0.9260	0.0248	0.9565	0.0193
0.6	0.9300	0.0249	0.9535	0.0195
0.7	0.9375	0.0247	0.9545	0.0196
0.8	0.9355	0.0248	0.9540	0.0195
0.9	0.9275	0.0248	0.9560	0.0195

6 Empirical applications

6.1 The demand for a risky asset in the presence of a background risk

In this example we consider an agent who is an S&P 500 Index tracker fund manager. R_s is the return of S&P 500 Index. Her income is $a + bR_s$, where $a \geq 0$ and $b > 0$ are constants. She has to allocate a sure wealth w between a safe asset paying a return r_f and Merrill Lynch US Corp AAA (or BBB) Total Return Index Value bond index tracker fund paying a random return R_b. She wants to choose $\theta \in [0, 1]$ to maximize expected utility. The problem can be written in the following compact manner

$$\max_{\theta} V(\theta) = E(u(w_0 + \theta X + a + bR_s)), \tag{18}$$

where $w_0 = w(1 + r_f)$ and $X = R_b - r_f$.

We employ a financial market index data set (available on the Federal Reserve Bank of St.Louis website, http://research.stlouisfed.org/fred2/) to estimate confidence bands. This data consists of weekly time serials: return of Merrill Lynch US Corp AAA Total Return Index Value, return of Merrill Lynch US Corp BBB Total Return Index Value , and return of S&P 500 Index, from 2006-02-11 to 2015-02-20. The Pearson's correlation coefficient and Kendall's tau for (AAA, S&P 500) and (BBB, S&P 500) are 0.5674, 0.3573(AAA); 0.6580, 0.4639(BBB) respectively. The scatter plots for this data set is also presented in Figure 1. Positive dependence between AAA(BBB) and S&P 500 are found. BBB and S&P 500 has larger positive dependence. However, from the Figure 1, their dependence structure is not linear.

We assume $a = 0$, $b = 1$ and $r_f = 0$ (a, b and r_f can be adjusted). We want to find I and J of $V'(0)$ for CARA and CRRA utility functions. The 95% two-sided confidence intervals for $V'(0)$ with different (λ, w_0) for CARA and (γ, w_0) for CRRA are presented in Table 2. From this table, we can have the following findings. First, all confidence intervals do not contain zero and thus we can conclude that $V'(0) > 0$ for these different (λ, w_0) and (γ, w_0). This implies that AAA and (or) BBB will be purchased. Second, for the same w_0, smaller absolute risk aversion coefficient (λ) or relative risk aversion coefficient (γ) can lead to larger $V'(0)$. As we mentioned before, $V'(0)$ represents the marginal expected utility for purchasing the first unit of a risky asset. So we can say, the lower the degree of risk aversion, the higher the marginal expected utility for purchasing the first unit of a risky asset. On the other hand, if we fix λ or γ, smaller w_0 also generally results in larger $V'(0)$. Hence, we conclude $V'(0)$ is decreasing in wealth. Finally, we find the values of $V'(0)$ for BBB are generally larger than those for AAA under the same settings. Therefore, in the sense of marginal expected utility for purchasing the first unit of a risky asset, the agent should purchase BBB rather than AAA.

By using Zhu et al. (2015)'s test statistics, the p-values for $H_0 : ED(y) \geq 0 \quad for\ all\ y$ are both almost 1 for AAA and BBB; the p-values for $H_0 : ED(y) \leq 0 \quad for\ all\ y$ are both 0 for AAA and BBB. Thus we cannot reject $H_0 : ED(y) \geq 0 \quad for\ all\ y$ and there exists some y, such that $ED(y) > 0$. Note that for CARA and CRRA utility functions, $u_{12} < 0$ and thus the integral $\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0, Y)F_Y(y)dy$ is negative. However, from the confidence intervals obtained in Table 2, $V'(0)$ can be safely asserted to be positive. As explained before, $V'(0)$ is the sum of a positive value of $E(u_1(w_0,Y))E(X)$ and the integral $\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0,Y)F_Y(y)dy$. As long as $E(u_1(w_0,Y))E(X) > -\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w_0,Y)F_Y(y)dy$, $V'(0) > 0$. This example clearly illustrate the necessity to construct confidence interval to determine the sign of $V'(0)$.

Table 2: The 95% two-sided confidence intervals for $V'(0)$ with different (λ, w_0) and (γ, w_0).

	$J(AAA)$	$I(AAA)$	$J(BBB)$	$I(BBB)$
(λ, w_0)(0.001,2000)	0.0158	0.0173	0.0172	0.0197
(0.001,1000)	0.0430	0.0469	0.0468	0.0536
(0.001,500)	0.0708	0.0774	0.0772	0.0884
(0.001,200)	0.0956	0.1044	0.1042	0.1193
(0.001,100)	0.1057	0.1154	0.1151	0.1318
(0.0005,2000)	0.0428	0.0447	0.0478	0.0510
(0.0005,1000)	0.0706	0.0737	0.0788	0.0841
(0.0005,500)	0.0907	0.0947	0.1012	0.1080
(0.0005,200)	0.1054	0.1100	0.1176	0.1255
(0.0005,100)	0.1108	0.1156	0.1236	0.1319
(γ, w_0) (1,2000)	0.1387	0.1424	0.1561	0.1623
(1,1000)	0.1968	0.2045	0.2202	0.2333
(1,500)	0.2495	0.2624	0.2775	0.2996
(1,200)	0.2975	0.3170	0.3289	0.3623
(1,100)	0.3180	0.3410	0.3506	0.3899
(0.5,2000)	8.1030	8.2075	9.1802	9.3586
(0.5,1000)	9.6447	9.8261	10.8961	11.2058
(0.5,500)	10.8475	11.1144	12.2219	12.6778
(0.5,200)	11.8346	12.1949	13.2991	13.9143
(0.5,100)	12.2304	12.6357	13.7276	14.4197

6.2 First-order risk aversion

We consider a DM with utility function $u(\cdot)$ who faces two potential monetary losses L_1 (losses to buildings) and L_2 (losses to buildings' contents or profit). We assume that only L_1 can be insured.

We use a widely studied Danish fire insurance data set (http://www.ma.hw.ac.uk/∼mcneil/data.html) to test ED. This data set contains 2167 fire insurance claims registered in Denmark in the years 1980-1990. The claims refer to loss to industrial dwellings and consist of loss to buildings (B), loss to their content (C) and loss to profit they generated (P). Follow Gijbels and Sznajder (2013), only positive claims in all three variables are considered. This reduces the sample size to 517. The Pearson's correlation coefficient and Kendall's tau for (B, C) and (B, P) are 0.6269, 0.1172(C); 0.7910, 0.2009(P) respectively. The scatter plots for this data set is also presented in Figure 1. Positive dependence between $C(P)$ and B are found. However, from the Figure 1, their dependence structure is not linear.

By using Zhu et al. (2015)'s test statistics, the p-values for $H_0 : ED(y) \geq 0 \quad for\ all\ y$ are 0.946 and 0.930 for C and P; the p-values for $H_0 : ED(y) \leq 0 \quad for\ all\ y$ are 0 and 0.004 for C and P respectively. Thus we cannot reject $H_0 : ED(y) \geq 0 \quad for\ all\ y$ and there exists some y, such that $ED(y) > 0$. Note that for CARA and CRRA utility functions, $u_{12} < 0$ and thus $\pi_{cd}(m) = -m\frac{\int_{\underline{y}}^{\bar{y}} ED(y)u_{12}(w,\tilde{y})F_Y(y)dy}{E(u_1(w,Y))}$ should be positive. However, is this term statistically significant positive? To answer this question, confidence interval shown in Proposition 4.7 is required.

We first set $X = B$ and $Y = C$. The confidence intervals of $\pi_{cd}(m)$ for CARA with $(\lambda, w_0) = (0.0005, 500), (0.001, 1000), (0.005, 5000)$ and $(0.01, 10000)$ are $m(-0.6974, 0.7234)$, $m(-1.3515, 1.4025)$, $m(-5.3188, 5.5352)$, and $m(-8.1329, 8.4940)$ respectively. The confidence intervals of $\pi_{cd}(m)$ for CRRA with $(\gamma, w_0) = (0.5, 1000), (0.5, 5000), (0.5, 10000), (1, 1000), (1, 5000)$ and $(1, 10000)$ are $m(-0.6561, 0.6810)$, $m(-0.1412, 0.1465)$, $m(-0.0713, 0.0739)$, $m(-1.2740, 1.3229)$, $m(-0.2807, 0.2911)$ and $m(-0.1421, 0.1474)$, respectively.

We then set $X = B$ and $Y = P$. The confidence intervals of $\pi_{cd}(m)$ for CARA with $(\lambda, w_0) = (0.0005, 500), (0.001, 1000), (0.005, 5000)$ and $(0.01, 10000)$ are $m(-0.3327, 0.3451)$, $m(-0.6556, 0.6800)$, $m(-2.9160, 3.0260)$, and $m(-5.0739, 5.2686)$ respectively. The confidence intervals of $\pi_{cd}(m)$ for CRRA with $(\gamma, w_0) = (0.5, 1000), (0.5, 5000), (0.5, 10000), (1, 1000), (1, 5000)$ and $(1, 10000)$ are $m(-0.3230, 0.3350)$, $m(-0.0669, 0.0694)$, $m(-0.0336, 0.0349)$, $m(-0.6366, 0.6604)$, $m(-0.1335, 0.1384)$ and $m(-0.0671, 0.0696)$, respectively.

Note that all above confidence intervals of $\pi_{cd}(m)$ contain zero and thus we cannot say that $\pi_{cd}(m)$ is statistically significant positive. To see this point clearer, look at Figure 2. In this Figure, the confidence bands for $ED(y)$ are presented. The central line in each subplot represents the empirical estimator of $ED(y)$. From this figure, we can see clearly that the $ED(y)$ curves in this study are very small. Although, there are some y satisfying $ED(y) > 0$, the curves are close to zero. This can finally make $\pi_{cd}(m)$ not statistically significant positive. From this study, we can also know that the test statistics proposed by Zhu et al. (2015) can only tell us whether $ED(y) \geq 0$ hold, but can not inform us the extent of $ED(y)$. For the later information, we need to look at the confidence band for $ED(y)$.

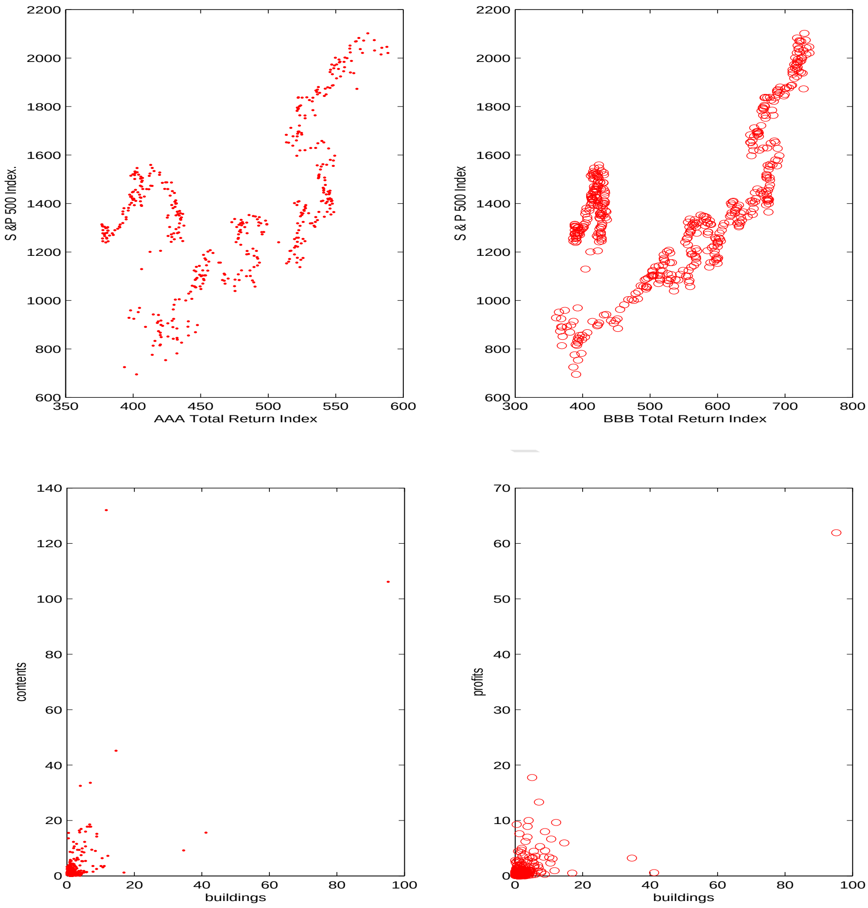

Figure 1: The scatter plots for studies in subsection 6.1 and 6.2 respectively.

7 Conclusion

ED is a key concept in many economics and finance studies. To conduct such studies, one need to precisely measure ED. The main contribution of this paper is to construct confidence bands for ED. We provide two examples to illustrate the easiness of implementing the proposed method

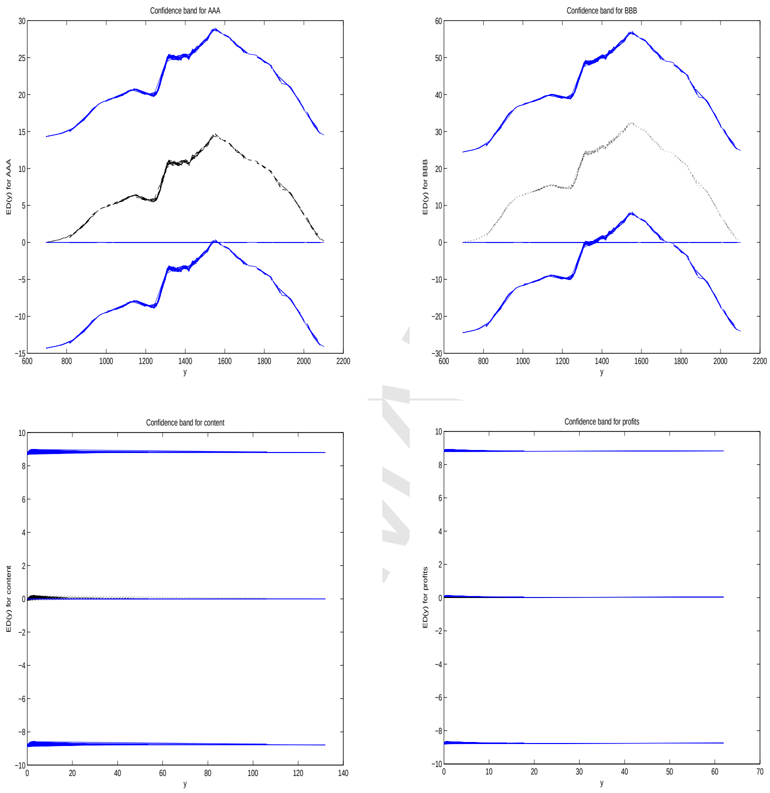

Figure 2: The confidence bands of ED(y)s for studies in subsection 6.1 and 6.2 respectively.

in practice. We also conduct a simulation study. Two empirical application are presented.

8 Appendix

8.1 Proof of Lemma 3.1

We first recall a result of Hoeffding's inequality.

Theorem 8.1 *(Hoeffding 1963, Theorem 2) If $X_1, X_2, \cdots, X_n$ are independent and $a_i \leq X_i \leq b_i$ $(i = 1, 2, \cdots, n)$, then for $t > 0$*

$$Pr\left(\bar{X} - E(\bar{X}) > t\right) \leq e^{-\frac{2n^2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}}. \tag{19}$$

It is obvious that $l_1(y), l_2(y), \cdots, l_n(y)$ are independent. For $y > \bar{y} \geq Y, I(Y_i \geq y) \equiv 0$ and thus $l_i(y) \equiv 0$. For $y < \underline{y} \leq Y, I(Y_i \geq y) \equiv 1$ and thus $l_i(y) \equiv 0$. For $\underline{y} \leq y \leq \bar{y}, -1 \leq I(Y_i \geq y) - E(I(Y_i \geq y)) \leq 1$ and thus $l_i(y)$ can be bounded for any y. First notice that $|l_i(y)| = |X_i - E(X)||I(Y_i \geq y) - E(I(Y_i \geq y))| \leq |X_i - E(X)|$. Recall that $\underline{x} \leq X_i \leq \bar{x}$ and thus $\underline{x} - E(X) \leq X_i - E(X) \leq \bar{x} - E(X)$. Thus $|X_i - E(X)| \leq \max\{E(X) - \underline{x}, \bar{x} - E(X)\}$. First if $E(X) - \underline{x} > \bar{x} - E(X)$, then $|l_i(y)| \leq |X_i - E(X)| \leq E(X) - \underline{x}$. Thus $y \in R, P(l_i(y) \in [\underline{x} - E(X), E(X) - \underline{x}]) = 1$. On the other hand, if $\bar{x} - E(X) > E(X) - \underline{x}$, then $|l_i(y)| \leq |X_i - E(X)| \leq \bar{x} - E(X)$. Thus $y \in R, P(l_i(y) \in [E(X) - \bar{x}, \bar{x} - E(X)]) = 1$. In sum, $range\{l_i(y)\} \leq 2\max\{E(X) - \underline{x}, \bar{x} - E(X)\} = C$. For any $y \in R$, $P(l_i(y) \in [-C/2, C/2]) = 1$.

By Hoeffding's inequality, we can get for any fixed y,

$$Pr\left(|\frac{1}{n}\sum_{i=1}^{n} l_i(y) - PED(y)| > \epsilon\right) \leq 2\exp\left(-\frac{2n\epsilon^2}{C^2}\right).$$

Note that if $\sup_y |\frac{1}{n}\sum_{i=1}^{n} l_i(y) - PED(y)| > \epsilon$, then for any $\eta > 0$, there exists y_0, such that $|\frac{1}{n}\sum_{i=1}^{n} l_i(y_0) - PED(y)| > \sup_y |\frac{1}{n}\sum_{i=1}^{n} l_i(y) - PED(y)| - \eta > \epsilon - \eta$. This implies that

$$\begin{aligned} & Pr\left(\sup_y |\frac{1}{n}\sum_{i=1}^{n} l_i(y) - PED(y)| > \epsilon\right) \\ \leq \quad & Pr\left(|\frac{1}{n}\sum_{i=1}^{n} l_i(y_0) - PED(y)| > \epsilon - \eta\right) \leq 2\exp\left(-\frac{2n(\epsilon-\eta)^2}{C^2}\right). \end{aligned}$$

The above inequality holds for any η. Then let $\eta \Rightarrow 0$ and the result is obtained.

8.2 Proof of Proposition 4.2

Define

$$L_F = \int_{\underline{y}}^{\bar{y}} U(y)u_{12}(w_0, y)dy \tag{20}$$

and

$$U_F = \int_{\underline{y}}^{\bar{y}} L(y) u_{12}(w_0, y) dy \tag{21}$$

From Propositions 3.2 and 4.1, we obtain

$$P\Big(E(u_1(w_0, Y))E(X) + L_F \leq V'(0) \leq E(u_1(w_0, Y))E(X) + U_F\Big) \geq 1 - \alpha. \tag{22}$$

Since

$$\begin{aligned}
L_F &= \int_{\underline{y}}^{\bar{y}} U(y) u_{12}(w_0, y) dy = \int_{\underline{y}}^{\bar{y}} [\widehat{PED}(y) + \epsilon_0] u_{12}(w_0, y) dy \\
&= \int_{\underline{y}}^{\bar{y}} \widehat{PED}(y) u_{12}(w_0, y) dy + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \\
&= \int_{\underline{y}}^{\bar{y}} \frac{1}{n} \sum_{i=1}^{n} \{X_i - \bar{X}\} I(Y_i \geq y) u_{12}(w_0, y) dy + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \\
&= \frac{1}{n} \sum_{i=1}^{n} \{X_i - \bar{X}\} \int_{\underline{y}}^{\bar{y}} I(Y_i \geq y) u_{12}(w_0, y) dy + \epsilon_{01} [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \\
&= \frac{1}{n} \sum_{i=1}^{n} \{X_i - \bar{X}\} \int_{\underline{y}}^{Y_i} u_{12}(w_0, y) dy + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \\
&= \frac{1}{n} \sum_{i=1}^{n} \{X_i - \bar{X}\} [u_1(w_0, Y_i) - u_1(w_0, \underline{y})] + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \\
&= \frac{1}{n} \sum_{i=1}^{n} \{X_i - \bar{X}\} u_1(w_0, Y_i) + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})], \qquad (23)
\end{aligned}$$

by defining

$$\begin{aligned}
J &= \frac{1}{n} \sum_{i=1}^{n} u_1(w_0, Y_i) \bar{X} + \frac{1}{n} \sum_{i=1}^{n} \{X_i - \bar{X}\} u_1(w_0, Y_i) + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \\
&= \frac{1}{n} \sum_{i=1}^{n} X_i u_1(w_0, Y_i) + \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})] \qquad (24)
\end{aligned}$$

and

$$I = \frac{1}{n} \sum_{i=1}^{n} X_i u_1(w_0, Y_i) - \epsilon_0 [u_1(w_0, \bar{y}) - u_1(w_0, \underline{y})], \tag{25}$$

we obtain

$$Pr\left(J \leq V'(0) \leq I\right) \geq 1 - \alpha, \quad as\ n \to \infty. \tag{26}$$

9 References

Denuit, M., Scaillet, O., 2004, Nonparametric tests for positive quadrant dependence. Journal of Financial Econometrics 2, 422-450.

Denuit, M., Huang, R. J., Tzeng, L. Y., 2015, Almost expectation and excess dependence notions. Theory and Decision, forthcoming.

Dhaene, J., Denuit, M., Vanduffel, S., 2009, Correlation order, merging and diversification. Insurance: Mathematics and Economics 45, 325-332.

Dionne, G., Li, J., 2014, When can expected utility handle first-order risk aversion? Journal of Economic Theory 154, 403-422.

Egozcue, M., Fuentes Garca, L. , Wong, W. K., Zitikis, R., 2011, Grss-type bounds for covariances and the notion of quadrant dependence in expectation, Central European Journal of Mathematics 9, 1288-1297.

Egozcue, M., Fuentes Garca, L. , Wong, W. K., Zitikis, R., 2013, Convex combinations of quadrant dependent copulas, Applied Mathematics Letters 26, 249-251.

Galton, F., 1886, Regression towards mediocrity in hereditary stature. Journal of the Anthropological Institute of Great Britain and Ireland 15, 246-263.

Gijbels, I., Omelka, M., Sznajder, D., 2010, Positive quadrant dependence tests for copulas. The Canadian Journal of Statistics 38(4), 555-581.

Gijbels, I., Sznajder, D., 2013, Positive quadrant dependence testing and constrained copula estimation. The Canadian Journal of Statistics 41(1), 36-64.

Hoeffding, W., 1963, Probability inequalities for sums of bounded random variables. Journal of the American Statistical Association 58, 13-30.

Hong, S. K. , Lew, K. O., MacMinn, R., Brockett, P., 2011, Mossins theorem given random initial wealth, Journal of Risk and Insurance 78, 309-324.

Johnson, N. L., Kotz, S., Balakrishnan, N., 1994, Continuous Univariate Distributions, Volume 1, Wiley.

Kallenberg, W. C. M., 2008, Modelling dependence. Insurance: Mathematics and Economics 42, 127-146.

Lehmann, E. L., 1966, Some concepts of dependence. Annals of Mathematical Statistics 37, 1137-1153.

Li, J., 2011, The demand for a risky asset in the presence of a background risk, Journal of Economic Theory, 146, 372-391.

Ledwina, T., Wylupek, G., 2014, Validation of positive quadrant dependence. Insurance: Mathematics and Economics 56, 38-47.

Loomes, G., Segal, U., 1994, Observing different orders of risk aversion, Journal of Risk and Uncertainty 9, 239-256.

Scaillet, O., 2005, A Kolmogorov-Smirnov type test for positive quadrant dependence. Canadian Journal of Statistics 33(3), 415-427.

Segal, U., Spivak, A., 1990, First order versus second order risk aversion, Journal of Economic Theory 51, 111-125.

Wong, K. P., 2013, Fixed versus variable rate loans under state-dependent preferences, Economic Modelling 31, 659-663.

Wong, K. P., 2012a, Production and futures hedging with state-dependent background risk. International Review of Economics & Finance 24, 177-184.

Wong, K. P., 2012b, Production and hedging under state-dependent preferences. Journal of Futures Markets 32, 945-963.

Wong, K. P., 2012c, Cross hedging with currency forward contracts. Journal of Futures Markets 33, 653-674.

Wong, K. P., 2014a, Hedging and the competitive firm under correlated price and background risk, Decisions in Economics and Finance 37, 329-340.

Wong, K. P., 2014b, Production and hedging in futures markets with multiple delivery specifications, Decisions in Economics and Finance 37, 413-421.

Wright, R., 1987, Expectation dependence of random variables, with an application in portfolio theory, Theory and Decision 22, 111-124.

Zhu, X. H., Guo, X., Lin, L., Zhu, L. X., 2015, Testing for positive expectation dependence, Annals of the institute of statistical mathematics, forthcoming.