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Irreducibility of Joint Inventory Positions in an Assemble-to-Order System under (r, nQ) Policies

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Abstract

In a typical assemble-to-order system, a customer order may request multiple items, and the order may not be filled if any of the requested items are out of stock. A key customer service measure when unfilled orders are backordered is the order-based backorder level. To evaluate this crucial performance measure, a fundamental question is whether the stationary joint inventory positions follow an independent and uniform distribution. In the present context, this is equivalent to the irreducibility of the Markov chain formed by the joint inventory positions. This paper presents a necessary and sufficient condition for the irreducibility of such a Markov chain through a set of simultaneous Diophantine equations. This result also leads to sufficient conditions that are more general than those in the literature.

Keywords: Order-based backorders; Assemble-to-order; Irreducibility; Diophantine equations; Necessary and sufficient condition; Unit-skeleton condition

1 Introduction

In a multi-item inventory system for assemble-to-order operations, a (customer) demand may request multiple items, and even multiple units of each item. An arriving demand cannot be filled if

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any requested item is not completely available. The steady-state level of order-based backorders, which counts the backorders of *demands* rather than of individual items, is a key measure of customer service performance for such systems when unfilled demands are backordered. Evaluating the average level of order-based backorders is hard. Given multiple item types in the same order, the amounts requested for individual items are mutually *dependent*, even if the arrivals of different types of demands are independent. Nevertheless, numerous settings exist in which the steady-state distribution of the joint inventory positions is independent and uniformly distributed, considerably simplifying the evaluation of the average level of order-based backorders, e.g., [6], [7], [2], [4].

Song [6] demonstrates that the joint inventory positions are uniformly distributed if and only if the corresponding Markov chain is irreducible. Song [6] then presents sufficient conditions for the irreducibility of the Markov chain based on customer *demand patterns* (i.e., the numbers of items requested by customer demands). These conditions are more general, and apply to more applications than another set of conditions proposed by Caplin [1] for a similar problem in a different context. See also Chen, Feng, and Simchi-Levi [2] for sufficient conditions specified by the customer demand patterns.

Since all the conditions found in the literature on the irreducibility of such Markov chains are sufficient in nature, one can ask the following questions: whether any new sufficient conditions exist that are easy to verify and yet more general than the known conditions; the impacts of order replenishment policies on the sufficient conditions; and, more importantly, whether a necessary and sufficient condition for a given system can be identified, and if so, whether it can be easily verified. This study answers these fundamental questions, which are useful in evaluating the average order-based backorders of assemble-to-order systems.

This study identifies a necessary and sufficient condition that incorporates the effects of both inventory replenishment policies and demand patterns. The condition is a set of algebraic relationships in the form of a set of *simultaneous linear Diophantine equations* (SLDE). Theorem 2 in Section 3 shows that the desired necessary and sufficient condition is equivalent to the existence of a feasible solution to every SLDE in the set. Example 1 shows how this necessary and sufficient condition can be used to verify the irreducibility of a Markov chain of the joint inventory positions, although none of the existing sufficient conditions in the literature can do so. (The

subsequent discussion uses some abbreviated expressions for compactness, for example, using “a system is irreducible” to mean “the Markov chain of the joint inventory positions of the system is irreducible”.) Section 4 first uses the algebraic properties of SLDE to show that two systems with different demand patterns can be simultaneously irreducible and hence equivalent. The procedure for showing this equivalence is called *demand filtering* in Proposition 1, and can be used to simplify the demand patterns of a system. The demand filtering procedure not only provides a simple explanation of the sufficiency of the conditions given in Song [6], but also leads to simple sufficient conditions that are more general. Example 3 in Section 4 demonstrates this result. This paper then proceeds with Proposition 2, which shows how a system with fewer item types in the demand pattern (i.e., *dimension reduction*) or lower order quantities (i.e., *quantity reduction*) can be identified such that the irreducibility of the simpler system implies that of the original system. See Example 4 in Section 4 for an illustration of these new ideas.

As a by-product of the main results, we determine the solution complexity of the set of SLDE in Theorem 3. Remark 1 further shows that the complexity can be of lower order than that of the standard algorithms for checking the irreducibility of a Markov chain based on its one-step transition probability matrix (pp 350-354 of Thulasiraman and Swamy [8]). The proposed procedure has another advantage: it does not require the use of the one-step transition probability matrix, which can be complicated to generate for multi-item inventory systems.

The remainder of this paper is organized as follows: Section 2 clarifies the relationship between the independent and uniformly distributed stationary distribution and the irreducibility of the Markov chain formed by the joint inventory positions. Section 3 gives the necessary and sufficient conditions for the chain to be irreducible. Section 4 gives sufficient conditions that are more general than the known ones. Examples are provided to illustrate these results. Finally, Section 5 presents conclusions.

2 The Stationary Distribution

Consider an inventory system involving I types of items. The system adopts the (r_i, nQ_i) inventory control policy for item $i, i = 1, \dots, I$: when the inventory position of item i is r_i or lower on review,

a replenishment order is issued requesting the minimum multiple of Q_i to move the inventory position of item i above r_i . Here, $Q_i \in Z^+$, where Z^+ denotes the set of positive integers, while $r_i \in Z$ and can be negative. The replenishment lead times are constant and may differ for different items.

There are N types of customer demands $\mathbf{d}_1, \dots, \mathbf{d}_j, \dots, \mathbf{d}_N$, which are generated by a random mechanism. For $j = 1, \dots, N$, the occurrence of demand $\mathbf{d}_j (\neq \mathbf{0})$ follows a Poisson process with rate $\lambda^{\mathbf{d}_j}$, independent of other types of demands. For all i, j , the demand $\mathbf{d}_j = (d_{1j}, \dots, d_{Ij})^T$ requires $d_{ij} \in Z^+ \cup \{0\}$ units of item i (where $(\cdot)^T$ is the transpose of (\cdot)).

Song [6] works with a similar demand process without any restriction on N . While the derivation below requires a finite N , the methods employed in the two papers are applicable to exactly the same demand processes. When the process used to verify the sufficient conditions in Song [6] is expressed as a detailed algorithmic procedure, which is necessary for problems of practical sizes, the worst-case computational effort of the procedure is a polynomial of N . Thus, N should be finite to meaningfully verify the conditions in Song [6]. Nevertheless, this does not mean that the proposed approaches, Song's and ours, only work for finite N . After elaborating the main results, this study explains in Remark 4, Section 4, that for a system with possibly an infinite number of demand types, it suffices to work with a finite subset of these demand types. The procedure for identifying such a finite subset of demand types is required in the sufficient conditions proposed in this study as well as the sufficient conditions proposed by Song. Thus, essentially no difference exists between the demand processes used in the two papers. For the time being, this study thus assumes finite N until Remark 4.

Let $\Theta = \{\mathbf{d}_1, \dots, \mathbf{d}_N\}$ denote the demand family, which is the collection of all demand types, and let $\mathbf{D} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_N)$ be the demand matrix. Moreover, let \mathbf{e}_i be an I -dimensional column vector of 1 at the i th element and 0 elsewhere, and let $\mathbf{A} = (Q_1\mathbf{e}_1, Q_2\mathbf{e}_2, \dots, Q_I\mathbf{e}_I)$ be the replenishment matrix that contains the re-order quantities Q_i of the I item types.

Let IP_i be the inventory position of item i . It is a classical result that the marginal stationary distribution of IP_i follows a uniform distribution when each request is for one unit of item i , or when requests are for multiple units of item i that are relatively prime to each other. The distribution of the joint inventory positions is complicated because the simultaneous requisitions of multiple types

induce dependence among the demand processes of different items.

Let $\mathbf{IP}(t) = (IP_1(t), \dots, IP_I(t))^T$ be the joint inventory positions at time t ; $\{\mathbf{IP}(t)\}$ is a continuous-time Markov chain (CTMC). Assume that the initial inventory position of item i takes a value in $\{r_i + 1, \dots, r_i + Q_i\}$. Let \mathcal{P} be the set of all possible joint inventory positions of $\{\mathbf{IP}(t)\}$. $\mathcal{P} = \{(IP_1, \dots, IP_I)^T : IP_i \in \{r_i + 1, \dots, r_i + Q_i\}, i = 1, \dots, I\}$ is a set of $\prod_{i=1}^I Q_i$ members. Moreover, let $\{\mathbf{IP}_m^-\}$ be the discrete-time Markov chain embedded at the demand arrival echoes of $\{\mathbf{IP}(t)\}$. $\{\mathbf{IP}_m^-\}$ has the same state space \mathcal{P} as $\{\mathbf{IP}(t)\}$ does. The following theorem shows that it is only necessary to discuss the properties of the embedded Markov chain $\{\mathbf{IP}_m^-\}$.

Theorem 1. *The unique stationary joint inventory position of $\{\mathbf{IP}(t)\}$ in \mathcal{P} is independent and uniformly distributed if and only if $\{\mathbf{IP}_m^-\}$ is irreducible. \square*

Theorem 1 follows easily from PASTA (Poisson Arrivals See Time Average) and the fact that $\{\mathbf{IP}_m^-\}$ is doubly stochastic. Song [6] obtains a similar result that requires both irreducibility and aperiodicity of $\{\mathbf{IP}_m^-\}$ to get the independent and uniformly distributed steady-state joint inventory positions. Since $\{\mathbf{IP}(t)\}$ is a CTMC with non-lattice sojourn times of states, the aperiodicity of the embedded chains $\{\mathbf{IP}_m^-\}$ is irrelevant to the existence of the stationary distribution of $\{\mathbf{IP}(t)\}$. Theorem 1 thus slightly relaxes Song's condition. Indeed a periodic, irreducible discrete-time $\{\mathbf{IP}_m^-\}$ can have an independent and uniform stationary distribution. See Remark 2 in Section 3 for more on this issue.

3 Necessary and Sufficient Condition

$\{\mathbf{IP}_m^-\}$ is irreducible if all the inventory positions in \mathcal{P} communicate with each other. This section expresses such an irreducible condition of $\{\mathbf{IP}_m^-\}$ as an algebraic relationship. The analysis applies to $Q_i > 1$. For $Q_i = 1$, the (r_i, nQ_i) policy reduces to the base-stock policy with a constant inventory position. Item i does not affect the irreducibility of $\{\mathbf{IP}_m^-\}$ and thus can be dropped from consideration. The algebraic relationship starts from the observation in Lemma 1 below, which basically expresses the accessibility between any pair of states in an irreducible chain as algebraic equations.

Lemma 1. $\{\mathbf{IP}_m^-\}$ is irreducible if and only if given any of its state $\mathbf{IP}^- = (IP_1^-, \dots, IP_i^-, \dots, IP_I^-)^T$, there exist non-negative $\mathbf{x} \in Z^N$, $\mathbf{x} \neq \mathbf{0}$, and non-negative $\mathbf{y} \in Z^I$ such that

$$\mathbf{IP}^- - \mathbf{D}\mathbf{x} + \mathbf{A}\mathbf{y} = \begin{cases} \mathbf{IP}^- + (Q_i - 1)\mathbf{e}_i, & \text{for any } IP_i^- = r_i + 1, \\ \mathbf{IP}^- - \mathbf{e}_i, & \text{for any } IP_i^- > r_i + 1. \end{cases} \quad (1)$$

□

Equation (1) can be interpreted naturally, as follows: Let \mathbf{x} denote the vector of the cumulative numbers of different types of requested demands (in a certain time interval) and let \mathbf{y} be the vector of the corresponding cumulative numbers of replenishment orders of the requested items. Suppose that there exist \mathbf{x} and \mathbf{y} that change the state from $(IP_1^-, \dots, IP_i^-, \dots, IP_I^-)^T$ to $(IP_1^-, \dots, r_i + Q_i, \dots, IP_I^-)^T$ for $IP_i^- = r_i + 1$, and to $(IP_1^-, \dots, IP_i^- - 1, \dots, IP_I^-)^T$ for any $IP_i^- > r_i + 1$. In this case, it is certainly possible to transform $\{\mathbf{IP}_m^-\}$ from one state to another by following this sequence of demands and replenishment orders. When such sequences of demands and replenishment orders exist for all pairs of states, $\{\mathbf{IP}_m^-\}$ is irreducible.

Equation (1) has integer coefficients and requires integer solutions; and is in fact a set of SLDE in number theory (see page 191 in Koshy [5]). The solution \mathbf{x} and \mathbf{y} are required to be non-negative in applications, creating a further constraint. Furthermore, to check irreducibility, Lemma 1 requires verifying relationship (1) for *all* states. This may create an astronomical number of sets of simultaneous equations to be verified in a typical application. Theorem 2 below reduces the number of sets of equations in the verification process to I and eliminates the non-negativity constraints on the solution.

Theorem 2. (The Necessary and Sufficient Condition) $\{\mathbf{IP}_m^-\}$ is irreducible if and only if each of the following I sets of SLDE has at least one pair of integer solutions $\mathbf{x} \in Z^N$ and $\mathbf{y} \in Z^I$:

$$\mathbf{A}\mathbf{y} - \mathbf{D}\mathbf{x} = -\mathbf{e}_i, \quad i = 1, \dots, I. \quad (2)$$

Proof. The set of equations (2) is a subset of (1). If $\{\mathbf{IP}_m^-\}$ is irreducible, (1) and hence (2) are satisfied according to Lemma 1. This establishes the necessity of the condition.

To demonstrate the sufficiency of the condition, first suppose there exists a non-negative integer solution (\mathbf{x}, \mathbf{y}) for each of the I sets of SLDE in (2). Then for $\mathbf{y}' = (y_1, \dots, y_{i-1}, y_i +$

$1, y_{i+1}, \dots, y_I)^T$,

$$\mathbf{A}\mathbf{y}' - \mathbf{D}\mathbf{x} = (Q_i - 1)\mathbf{e}_i. \quad (3)$$

Equations (2) and (3) correspond to the two equalities in (1), and hence establish the irreducibility of $\{\mathbf{IP}_m^-\}$ for the non-negative integer solution (\mathbf{x}, \mathbf{y}) .

Next suppose that x_j is negative for some j . There certainly exists some integer $a > 0$ such that $a \prod_{i=1}^I Q_i + x_j > 0$. Let $\mathbf{x}' = (x_1, \dots, x_{j-1}, x_j + a \prod_{i=1}^I Q_i, x_{j+1}, \dots, x_N)^T$ and $\mathbf{y}' = (y_1 + \frac{ad_{1j}}{Q_1} \prod_{i=1}^I Q_i, \dots, y_I + \frac{ad_{Ij}}{Q_I} \prod_{i=1}^I Q_i)^T$. Replacing \mathbf{y} and \mathbf{x} with \mathbf{y}' and \mathbf{x}' , respectively, yields $\mathbf{A}\mathbf{y}' - \mathbf{D}\mathbf{x}' = -\mathbf{e}_i$ with $x'_j > 0$. Following the same procedure, it is possible to construct $y'_j > 0$ and eventually obtain a non-negative solution to (3). \square

Next this study presents the computational complexity of verifying the condition in Theorem 2. Let $Q_{max} = \max\{Q_1, \dots, Q_N\}$ and suppose that $d_{ij} < Q_i$ for all i and j . Furthermore, let $\lceil(\cdot)\rceil$ denote the smallest integer no less than (\cdot) .

Theorem 3. *The complexity of verifying the necessary and sufficient condition for the irreducibility of $\{\mathbf{IP}_m^-\}$ is a polynomial in $\max\{I, N\}$, or more specifically, is of order $O(IK)$ where*

$$K = \rho^2(2I + N)(I + N)^3 + \rho I(2I + N), \text{ for } \rho = I + N + I\lceil\log(1 + I \times Q_{max})\rceil.$$

Proof. Using Theorem 13 of Chou and Collins [3], finding a feasible solution for one set of SLDE is of order $O(K)$. This theorem follows from solving I sets of SLDE as in Theorem 2. \square

Remark 1. (a) *With N types of demand and I types of items, the Markov chain of joint inventory positions has $\prod_{i=1}^I Q_i$ states and $N \prod_{i=1}^I Q_i$ different types of transitions. Based on Thulasiraman and Swamy (pp 350-354 of [8]), the complexity of verifying the irreducibility of such a Markov chain is $O(N \prod_{i=1}^I Q_i + \prod_{i=1}^I Q_i)$, i.e., $O(N \prod_{i=1}^I Q_i)$, given the one-step transition matrix. From Theorem 3, the computational complexity of verifying the set of SLDE is $O(N^6 I^7)$. Generally, the relative merits of the two approaches in terms of complexity depend on the system parameters.*

(b) *The irreducibility of the Markov chain remains unchanged if any d_{ij} of \mathbf{d}_j is replaced by $(d_{ij} \bmod Q_i)$. For $d_{ij} \geq Q_i$, simply set $d_{ij} = (d_{ij} \bmod Q_i)$ before applying Theorem 2. The*

maximum number of such modulus operations is NI . Theorem 3 remains valid because the modulus operation has a light computational workload compared to the operations in the theorem.

Example 1. Consider a three-item system with demand family $\Theta = \{(1, 1, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T\}$, and replenishment order policy $(r_i, Q_i) = (0, 3)$ for items $i = 1, 2, 3$. The literature contains no sufficient condition identifying the irreducibility of $\{\mathbf{IP}_m^-\}$ for this system. By applying Theorem 2 the set of SLDE $\mathbf{Ay} - \mathbf{Dx} = -\mathbf{e}_1$ is

$$\begin{cases} 3y_1 - x_1 - x_3 = -1, \\ 3y_2 - x_1 - x_2 = 0, \\ 3y_3 - x_2 - x_3 = 0. \end{cases}$$

Applying the algorithm in Chou and Collins [3] obtains an integer solution $(x_1, x_2, x_3) = (-1, 1, 2)$ and $(y_1, y_2, y_3) = (0, 0, 1)$ for $\mathbf{Ay} - \mathbf{Dx} = -\mathbf{e}_1$. Following the procedure that proves Theorem 2, this solution can be transformed to the non-negative solution $(x_1, x_2, x_3) = (26, 1, 2)$ and $(y_1, y_2, y_3) = (9, 9, 1)$. The symmetry of the item parameters indicates the existence of at least one integer solution for $\mathbf{Ay} - \mathbf{Dx} = -\mathbf{e}_2$ or $\mathbf{Ay} - \mathbf{Dx} = -\mathbf{e}_3$. Hence, $\{\mathbf{IP}_m^-\}$ is irreducible. \square

Example 2. (A reducible $\{\mathbf{IP}_m^-\}$) Consider the demand family from Example 1 but different replenishment order policies $(r_i, Q_i) = (0, 2)$ for items $i = 1, 2, 3$. From Theorem 2, the SLDE $\mathbf{Ay} - \mathbf{Dx} = -\mathbf{e}_1$ comprises the set of equations

$$\begin{cases} 2y_1 - x_1 - x_3 = -1, \\ 2y_2 - x_1 - x_2 = 0, \\ 2y_3 - x_2 - x_3 = 0. \end{cases}$$

The last two equations gives $x_1 - x_3 = 2y_2 - 2y_3$, i.e., $x_1 - x_3$ is an even number, including zero. The result implies that $x_1 + x_3$ is an even number, contradicting the first equation. Thus, the SLDE has no solution, and $\{\mathbf{IP}_m^-\}$ is reducible.

The result can be deduced intuitively from the nature of the system. Independent of type, a customer demand reduces the sum of the inventory positions of all items (i.e. $IP_1^- + IP_2^- + IP_3^-$ in this case) by two and a replenishment order increases the sum by two. If the sum of the inventory positions of all items is an even number initially, it remains that way continuously. Similarly, the sum stays odd if it starts that way. This shows that the chain is reducible. \square

The above two examples show that the existing sufficient conditions have their limitations and that the replenishment policies can affect the irreducibility.

Remark 2. *Indeed it is possible to have a multi-item, periodic, and irreducible $\{\mathbf{IP}_m^-\}$ whose steady-state distribution is independent and uniformly distributed. Consider a three-item system where $\Theta = \{(1, 1, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T\}$ and $(r_i, Q_i) = (0, 3)$ for items $i = 1, 2, 3$, as in Example 1. Based on the argument in Example 1, $\{\mathbf{IP}_m^-\}$ is irreducible so that the independent and uniformly distributed stationary distribution of the joint inventory positions follows from Theorem 1. A simple argument regarding the possible number of transitions required to return to state $(3, 3, 3)$ shows that $\{\mathbf{IP}_m^-\}$ is periodic of period 3. Of course the periodicity affects only $\{\mathbf{IP}_m^-\}$ and has no effect on the stationary distribution of $\{\mathbf{IP}(t)\}$. \square*

4 Sufficient Conditions Derived from Simplification Procedures

While its computation is polynomially bounded, verifying the necessary and sufficient condition may still take time. This section provides new and efficient procedures to verify the irreducibility of $\{\mathbf{IP}_m^-\}$. The main results obtained in this section are Propositions 1 and 2. As discussed in the Introduction, Demand Filtering in Proposition 1 gives the condition for two systems with different demand patterns to be irreducible together. Furthermore, the Dimension Reduction and Quantity Reduction in Proposition 2 provide sufficient conditions for the irreducibility of a system when a simpler system is irreducible.

Recall that \mathbf{e}_i can represent a unit- i demand. If $\{\mathbf{e}_1, \dots, \mathbf{e}_I\} \subseteq \Theta$, $\{\mathbf{IP}_m^-\}$ is clearly irreducible. $\{\mathbf{e}_1, \dots, \mathbf{e}_I\} \subseteq \Theta$ was the first sufficient condition given in Song [6] for the irreducibility of $\{\mathbf{IP}_m^-\}$, and was also the first sufficient condition for the problem. For ease of reference, this condition is here termed the *unit-skeleton condition* in this paper. Furthermore, let

$$IP^-(\Theta, \mathbf{Q}) = \begin{cases} 1, & \text{if } \{\mathbf{IP}_m^-\} \text{ is irreducible under the demand family } \Theta \text{ and replenishment } \mathbf{Q}, \\ 0, & \text{otherwise,} \end{cases}$$

where $IP^-(\emptyset, [\]) = 1$. For a system with demand family Θ and replenishment in multiples of \mathbf{Q} , Propositions 1 and 2 below will transform Θ , \mathbf{Q} , or both Θ and \mathbf{Q} , to a simpler demand family Φ , possibly with a new replenishment vector \mathbf{Q}_{new} . The transformation maintains

$IP^-(\Theta, \mathbf{Q}) = IP^-(\Phi, \mathbf{Q}_{new})$ throughout, i.e., the embedded chains $\{\mathbf{IP}_m^-\}$ of the old and new systems are simultaneously either irreducible or reducible. After a round of transformation, if $\{\mathbf{e}_1, \dots, \mathbf{e}_I\}$ is a subset of the demand family Φ of the transformed system, then the transformed system satisfies the unit-skeleton condition and the original system is irreducible.

The transformation procedures in Propositions 1 and 2 can be applied iteratively and repeatedly until either the resultant system is found to be unit-skeleton, or it can no longer be further simplified and does not satisfy the unit-skeleton condition. In the latter case, the procedures presented in this section fail to determine the irreducibility of $\{\mathbf{IP}_m^-\}$, making it necessary to apply the necessary and sufficient condition from the previous section to determine the irreducibility.

Demand Filtering in Proposition 1 below is a transformation of demands and the demand family Θ . For demands $\{\mathbf{d}_j, \mathbf{d}_k\} \subseteq \Theta$ such that $\mathbf{d}_j \geq \mathbf{d}_k$, the larger demand \mathbf{d}_j is *filtered out* and is replaced by demand $\mathbf{d}_j - \mathbf{d}_k$ to create a new demand family Φ .

Proposition 1. (Demand Filtering) *Suppose that $\{\mathbf{d}_j, \mathbf{d}_k\} \subseteq \Theta$ such that $d_{ik} \geq d_{ij}$ for any i and $\mathbf{d}_j \neq \mathbf{d}_k$. Then $IP^-(\Theta, \mathbf{Q}) \equiv IP^-(\Phi, \mathbf{Q})$ where $\Phi = \Theta - \{\mathbf{d}_k\} + \{\mathbf{d}_k - \mathbf{d}_j\}$.*

Proof. Without the loss of generality, suppose $j < k$. For any $i \in \{1, 2, \dots, I\}$, $\mathbf{A}\mathbf{y} - (\mathbf{d}_1, \dots, \mathbf{d}_j, \dots, \mathbf{d}_k, \dots, \mathbf{d}_N) \times (x_1, \dots, x_j, \dots, x_k, \dots, x_I)^T = -\mathbf{e}_i$ has at least one integer solution if and only if $\mathbf{A}\mathbf{y} - (\mathbf{d}_1, \dots, \mathbf{d}_j, \dots, \mathbf{d}_k - \mathbf{d}_j, \dots, \mathbf{d}_N) \times (x_1, \dots, x_j + x_k, \dots, x_k, \dots, x_I)^T = -\mathbf{e}_i$ has at least one integer solution. The result then follows from Theorem 2. \square

Remark 3. *Song [6] gives two other sufficient conditions **S2** and **S3** in addition to $\{\mathbf{e}_1, \dots, \mathbf{e}_I\} \subseteq \Theta$. Please refer to the appendix for the details of **S2** and **S3**. Applying Proposition 1, it is easy to show that **S2** is sufficient for **S3**, and **S3** is in turn sufficient for the unit-skeleton condition. Thus, Proposition 1 unifies the three sufficient conditions of Song [6]. As reflected from Example 3 below, the proposition is in fact more general than Song's conditions.*

Example 3. *Let $\Theta = \{(1, 3)^T, (2, 7)^T\}$. Notice that Θ does not satisfy any of the three conditions in Song [6]. Applying Proposition 1 once gives $\{(1, 3)^T, (1, 4)^T\}$, and once again gives $\Phi = \{(1, 3)^T, (0, 1)^T\}$. Repeatedly applying Proposition 1 to reduce item 2 of the first demand gives the unit-skeleton demand family $\Phi' = \{(1, 0)^T, (0, 1)^T\}$, Thus, $IP^-(\Theta, \mathbf{Q}) = IP^-(\Phi, \mathbf{Q}) = 1$. \square*

Demand Filtering in Proposition 1 shows that two systems of different demand patterns can

be irreducible together. Each step of Demand Filtering considers only two types of demands. The rest of this section discusses related but different results. Given the demand pattern Θ and the order quantities \mathbf{Q} of a system, Proposition 2 presents sufficient conditions for the existence of a system with possibly fewer demand types in its demand pattern Φ and lower order quantities \mathbf{Q}_{new} (i.e., $Q_{i,new} \leq Q_i$ if item i remains in Φ) than the original system such that the irreducibility of the simpler system *implies* the irreducibility of the original system. These sufficient conditions consider the relationship among multiple types of demands and their order quantities, a feature not discussed in the literature.

The derivation relies on the simplification of the SLDE in Equation (2). This study gives sufficient conditions such that the existence of solutions to the SLDE of a simpler system implies the existence of solutions to the SLDE of the original system. Two procedures are considered in our sufficient conditions. Demand Reduction identifies sufficient conditions to drop items in the demand pattern while Quantity Reduction reduces the order quantities of items. Both procedures reduce the size of the state space of the joint inventory positions. The idea of each procedure is first illustrated by a simple example focusing on item 1 for a system of three items and (incidentally) of three demand types. These two procedures are then combined in Proposition 2 as one general result.

The simplification of SLDE involves concepts in Theorem 4.2 of Koshy [5], which are formulated as Lemma 2 below. The lemma and subsequent derivation require the *greatest common divisor* of a collection of integers a_1, \dots, a_m , denoted as $gcd(a_1, \dots, a_m)$. By convention, $gcd(a_1, 0)$ for $a_1 > 0$ denotes the value of a_1 , i.e., $gcd(a_1, 0) = a_1$ in applications. Similarly, $lcm(a_1, \dots, a_m)$ denotes the corresponding *least common multiple* of the integers a_1, \dots, a_m .

Lemma 2. *Let a_1, \dots, a_m and c be integers. The linear Diophantine equation*

$$a_1 z_1 + a_2 z_2 + \dots + a_m z_m = c$$

has an integer solution for z_1, \dots, z_m iff c is divisible by $gcd(a_1, \dots, a_m)$. When the equation is solvable, it has an infinite number of solutions. □

Dimension Reduction

From Theorem 2, $IP^-(\Theta, \mathbf{Q}) = 1$ iff there are integer solutions of $y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}$, and $x_3^{(i)}$ to

$$\begin{pmatrix} Q_1 y_1^{(i)} \\ Q_2 y_2^{(i)} \\ Q_3 y_3^{(i)} \end{pmatrix} - \begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \end{pmatrix} x_1^{(i)} - \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \end{pmatrix} x_2^{(i)} - \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \end{pmatrix} x_3^{(i)} = -\mathbf{e}_i, \quad i = 1, 2, 3. \quad (4)$$

Notably, for clarity, superscripts “ (i) ” are added to the solutions of SLDE (2).

Now, consider another system involving two items, items 2 and 3, and two demand types that request the same quantities of items 2 and 3 as in the original three-item system. This reduced system is irreducible if and only if there exist integer solutions of $y_2^{(i)}, y_3^{(i)}, x_2^{(i)}, x_3^{(i)}$ to

$$\begin{pmatrix} Q_2 y_2^{(i)} \\ Q_3 y_3^{(i)} \end{pmatrix} - \begin{pmatrix} d_{22} \\ d_{32} \end{pmatrix} x_2^{(i)} - \begin{pmatrix} d_{23} \\ d_{33} \end{pmatrix} x_3^{(i)} = - \begin{pmatrix} 3 - i \\ i - 2 \end{pmatrix}, \quad i = 2, 3. \quad (5)$$

Notably, \mathbf{e}_i in (4) are 3-dimensional column vectors and the right-hand-side of (5) are 2-dimensional column vectors. Further suppose that there exists an integer solution $y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, x_1^{(1)}$ for the following equations

$$\begin{pmatrix} Q_1 y_1^{(1)} \\ Q_2 y_2^{(1)} \\ Q_3 y_3^{(1)} \end{pmatrix} - \begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \end{pmatrix} x_1^{(1)} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \quad (6)$$

It is possible to construct integer solutions of (4) from the integer solutions of (5) and (6). Check that $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, x_1^{(1)}, 0, 0)$ satisfies (4) for $i = 1$; $(-y_1^{(1)}(d_{12}x_2^{(2)} + d_{13}x_3^{(2)}), -y_2^{(1)}(d_{12}x_2^{(2)} + d_{13}x_3^{(2)}) + y_2^{(2)}, -y_3^{(1)}(d_{12}x_2^{(2)} + d_{13}x_3^{(2)}) + y_3^{(2)}, -x_1^{(1)}(d_{12}x_2^{(2)} + d_{13}x_3^{(2)}), x_2^{(2)}, x_3^{(2)})$ satisfies (4) for $i = 2$; and $(-y_1^{(1)}(d_{12}x_2^{(3)} + d_{13}x_3^{(3)}), -y_2^{(1)}(d_{12}x_2^{(3)} + d_{13}x_3^{(3)}) + y_2^{(3)}, -y_3^{(1)}(d_{12}x_2^{(3)} + d_{13}x_3^{(3)}) + y_3^{(3)}, -x_1^{(1)}(d_{12}x_2^{(3)} + d_{13}x_3^{(3)}), x_2^{(3)}, x_3^{(3)})$ satisfies (4) for $i = 3$. In other words, if there exists an integer solution for equations (6), then the existence of integer solutions for (5) implies the same for (4). Thus, given an integer solution for (6), a simpler system can be considered where the demand pattern excludes item 1. The following deduces sufficient conditions for (6) to hold.

First suppose that $d_{21}, d_{31} > 0$. For (6) to have an integer solution, x_1 must satisfy $Q_i y_i = d_{i1} x_1$, $i = 2, 3$. Simplifying, $x_1 = \frac{Q_i y_i}{d_{i1}} = \frac{Q_i}{\gcd(Q_i, d_{i1})} \frac{y_i \gcd(Q_i, d_{i1})}{d_{i1}}$, i.e., x_1 is a multiple of $\frac{Q_i}{\gcd(Q_i, d_{i1})}$.

Any solution of x_1 that satisfies both $Q_i y_i = d_{i1} x_1, i = 2, 3$, must be a multiple of $lcm(\frac{Q_2}{gcd(Q_2, d_{21})}, \frac{Q_3}{gcd(Q_3, d_{31})})$. Thus, (6) and the following equation (7) have an integer solution simultaneously.

$$Q_1 y_1 - d_{11} \times lcm(\frac{Q_2}{gcd(Q_2, d_{21})}, \frac{Q_3}{gcd(Q_3, d_{31})}) x_1 = -1. \quad (7)$$

To deduce a condition for the existence of an integer solution in (7), and hence in (6), Lemma 2 provides the sufficient condition:

$$\Omega(1, \mathbf{d}_1) \equiv gcd(Q_1, d_{11} \times lcm(\frac{Q_2}{gcd(Q_2, d_{21})}, \frac{Q_3}{gcd(Q_3, d_{31})})) = 1, \quad (8)$$

where the two parameters of $\Omega(\cdot, \cdot)$ are for the item and the demand under consideration, respectively. When (8) holds, (7) follows; and so does (6). The existence of a solution of (5) then implies that of (4), i.e., the irreducibility of the reduced system with items 2 and 3 implies the irreducibility of the original system with items 1, 2, and 3.

Similarly, it can be concluded that the sufficient condition (8) for (6) remains applicable when $d_{21} = 0$ or $d_{31} = 0$. Notably, $gcd(a_1, 0) = a_1$.

Quantity Reduction

This study tries to reduce the quantity of items required in a demand type, even if no demand type is eliminated from the demand pattern. Suppose

$$\Omega(1, \mathbf{d}_1) \equiv gcd(Q_1, d_{11} \times lcm(\frac{Q_2}{gcd(Q_2, d_{21})}, \frac{Q_3}{gcd(Q_3, d_{31})})) > 1. \quad (9)$$

It is impossible to use dimension reduction to find a reduced system without item 1. Instead, given the inventory positions of the items, it is sufficient to consider an equivalent system such that the requested and replenishment quantities of item 1 are no more than $\Omega(1, \mathbf{d}_1)$.

Based on (9) and the definition of $\Omega(1, \mathbf{d}_1)$, $1 < \Omega(1, \mathbf{d}_1) \leq Q_1$. If $\Omega(1, \mathbf{d}_1) = Q_1$, nothing can be done. The following only considers the case when $\Omega(1, \mathbf{d}_1) < Q_1$. Using Lemma 2, condition (9) implies that

$$Q_1 y_1 - d_{11} \times lcm(\frac{Q_2}{gcd(Q_2, d_{21})}, \frac{Q_3}{gcd(Q_3, d_{31})}) x_1 = \Omega(1, \mathbf{d}_1) \quad (10)$$

has an integer solution. Recall from (1) and (2) that the set of SLDE traces the changes in inventory positions induced by demand arrivals and replenishment orders. Equation (10) in-

dicates that for any given order quantities \mathbf{Q} , for item 1, the overall effect of the sequence of $lcm(\frac{Q_2}{gcd(Q_2, d_{21}), \frac{Q_3}{gcd(Q_3, d_{31})})x_1$ arrivals of \mathbf{d}_1 demands will change the inventory position of item 1 by $\Omega(1, \mathbf{d}_1)$, while simultaneously holding the inventory position of items 2 and 3 constant. This last point can be traced back to (6) where the inventory positions of items 2 and 3 are held fixed. Effectively there exists at least one sequence of demands \mathbf{d}_1 and replenishment orders of items 1 such that the inventory position IP_1 of item 1 is changed in *any* multiple of $\Omega(1, \mathbf{d}_1)$, while keeping the inventory positions of other items constant. As far as this sequence of \mathbf{d}_1 demands is concerned, the inventory position of item 1, IP_1 , is divided into $\Omega(1, \mathbf{d}_1)$ communication classes, where $(IP_1 \bmod \Omega(1, \mathbf{d}_1)) = k$ for class k , $k = 0, 1, \dots, \Omega(1, \mathbf{d}_1) - 1$. The inventory positions of item 1 communicate with each other if these $\Omega(1, \mathbf{d}_1)$ communication classes do so, where the latter is determined by all demands, not only the specific sequence of \mathbf{d}_1 demands that generates the $\Omega(1, \mathbf{d}_1)$ communication classes.

To consider the effect of all demands on item 1, the order quantities and the requested quantities of item 1 in various demand types are further changed. Notably, the reordering quantity Q_1 is a multiple of $\Omega(1, \mathbf{d}_1)$. Starting from an inventory position IP_1 of class k , the arrival of a replenishment order for item 1 preserves the class of the inventory position of that item. As far as the communication of these $\Omega(1, \mathbf{d}_1)$ classes of inventory positions of item 1 is concerned, the replenishment quantity Q_1 has the same effect as the replenishment quantity $\Omega(1, \mathbf{d}_1) < Q_1$. Thus, it is sufficient to consider a modified system with reduced reordering quantity $\Omega(1, \mathbf{d}_1)$ for item 1. Furthermore, demand \mathbf{d}_j changes the inventory position of item 1 by $(d_{1j} \bmod \Omega(1, \mathbf{d}_1))$ among these $\Omega(1, \mathbf{d}_1)$ communication classes. Therefore, it is acceptable to reduce d_{1j} to $(d_{1j} \bmod \Omega(1, \mathbf{d}_1))$ in checking the irreducibility of the modified system.

The General Result that Combines Dimension Reduction and Quantity Reduction

The above two procedures can be combined into one general result in Proposition 2 for any number of demands and any number of items. Define

$$\Omega(i, \mathbf{d}_j) \equiv gcd \left(Q_i, d_{ij} \times lcm \left(\frac{Q_1}{gcd(d_{1j}, Q_1)}, \dots, \frac{Q_{i-1}}{gcd(d_{i-1,j}, Q_{i-1})}, \frac{Q_{i+1}}{gcd(d_{i+1,j}, Q_{i+1})}, \dots, \frac{Q_I}{gcd(d_{Ij}, Q_I)} \right) \right).$$

Let $\mathbf{Q}^{-i} = (Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_I)$ and $\mathbf{d}_j^{-i} = (d_{1j}, \dots, d_{i-1,j}, d_{i+1,j}, \dots, d_{Ij})^T$ be, respectively, the

modification of \mathbf{Q}_i and \mathbf{d}_j with the information of item i dropped; and $\alpha(i, \mathbf{d}_j) = d_{ij} \bmod \Omega(i, \mathbf{d}_j)$.

Proposition 2. (Dimension/Quantity Reduction) *For a demand $\mathbf{d}_j \in \Theta$, the system can be simplified in the following situations:*

1. If $\Omega(i, \mathbf{d}_j) = 1$, $IP^-(\Theta, \mathbf{Q}) = IP^-(\Phi, \mathbf{Q}^{-i})$ where $\Phi = \{\mathbf{d}_1^{-i}, \dots, \mathbf{d}_N^{-i}\} - \{(0, \dots, 0)_{1 \times (I-1)}^T\}$.
2. If $1 < \Omega(i, \mathbf{d}_j) < Q_i$, $IP^-(\Theta, \mathbf{Q}) = IP^-(\Phi, \mathbf{Q}_{new})$ where $\Phi = \Theta - \{\mathbf{d}_j\} + \{\mathbf{d}_j - d_{ij}\mathbf{e}_i + \alpha(i, \mathbf{d}_j)\mathbf{e}_i\} - \{(0, \dots, 0)_{1 \times (I)}^T\}$ and $\mathbf{Q}_{new} = (Q_1, \dots, Q_{i-1}, \Omega(i, \mathbf{d}_j), Q_{i+1}, \dots, Q_I)$. \square

Example 4. Consider a system with $\Theta = \{(4, 2)^T, (3, 3)^T\}$ and $\mathbf{Q} = (8, 9)$. Applying Proposition 2 once to item 1, demand \mathbf{d}_1 , yields $\Omega(1, \mathbf{d}_1) = \gcd\left(8, 4 \times \frac{9}{\gcd(2, 9)}\right) = 4$ so that the new system has $\Phi = \{(0, 2)^T, (3, 3)^T\}$ and $\mathbf{Q}_{new} = (4, 9)$. Then consider item 2 and demand \mathbf{d}_1 . From $\Omega(2, \mathbf{d}_1) = \gcd\left(9, 2 \times \frac{4}{\gcd(0, 4)}\right) = 1$, item 2 can be dropped. The result is a system with only item 1 such that $\Phi = \{(3)\}$ and $\mathbf{Q}^{-2} = (4)$. Obviously, $IP^-(\Theta, \mathbf{Q}) \equiv IP^-(\Phi, \mathbf{Q}^{-2}) = 1$ since “3” and “4” are relative prime. \square

Remark 4. Changing demand components (i.e. d_{ij}) certainly affects the stochastic behavior of the Markov chains of joint inventory positions. However, the irreducibility of a Markov chain is determined only by the signs of the transition probabilities, not their exact values. For the same reason, whether the demand types are randomly generated is irrelevant.

For random demand arrivals, there certainly exist cases with infinitely many demand types. However, even for these cases, it is sufficient to work with $|\mathbf{D}|$ demand types, where $|\mathbf{D}| \leq \prod_{i=1}^I Q_i$. Note that the irreducibility of the Markov chain remains unchanged if any d_{ij} of \mathbf{d}_j is replaced by $(d_{ij} \bmod Q_i)$. For unbounded N , given Q_i , it is sufficient to modify the demand component d_{ij} of all the demands for items i by taking the modulus operation with respect to Q_i , leading to a maximum of Q_i types of demands for item i . Applying similar modulus operations to all types of items gives the upper bound $\prod_{i=1}^I Q_i$ on the total number of relevant demand types.

When the sufficient conditions in the literature spell out their steps algorithmically, they also become a function of N . Thus, when N is infinite, these sufficient conditions require an argument equivalent to ours in the above paragraph to work with a subset of all possible demand types.

5 Conclusions

The average level of the order-based backorders is an important performance measure for multi-item inventory systems. As shown in the literature, the evaluation of such a performance measure relies on the steady-state joint inventory positions being independent and uniformly distributed. This study algebraically expresses the necessary and sufficient conditions for the existence of such a steady-state distribution. Furthermore, this study provides more general sufficient conditions that can identify more applications than the sufficient conditions presented in the literature, as illustrated in Examples 3 and 4.

A natural question arises: For a system with irreducible inventory positions after a sequence of demand transformations, must the resulting system satisfy the unit skeleton condition? The answer is mixed, and basically rests on the definition of the unit skeleton condition. For example, the procedures in Propositions 1 - 2 fail to identify the irreducibility for the three-item system in Example 1. Suppose that the unit skeleton condition is defined equivalently to Lemma 1, i.e., complete freedom exists to change the inventory position of any item while fixing the inventory positions of other items. Then the unit skeleton condition is certainly also necessary under this definition.

Appendix. Sufficient Conditions in Song [6]

Let K_0 be a subset of $\{1, 2, \dots, I\}$ and \mathbf{d}_{K_0} be a demand type that requires every item in K_0 . The second and third sufficient conditions of Song are restated below.

Condition S2 *There exists a demand \mathbf{d}_{K_0} such that the following three requirements are satisfied simultaneously:*

1. *For any $i \in K_0$, there exists a demand in the demand family which either only requires item i or requires all the items in the set $K_0 - \{i\}$;*
2. *For any item $i \notin K_0$, there exists a demand in the demand family which requires all the items in the set $K_0 + \{i\}$; and*
3. *Unit demand is possible for all the above demand types.* □

In the next sufficient condition, the unit demand requirement is unnecessary.

Condition S3 *There exists a batch demand \mathbf{d}_{K_0} such that the following two requirements are satisfied simultaneously:*

1. *For any $i \in K_0$, either $\mathbf{d}_{K_0} + \mathbf{e}_i$ or $\mathbf{d}_{K_0} - \mathbf{e}_i$ is a demand type in the demand family; and*
2. *For any other item $i \notin K_0$, $\mathbf{d}_{K_0} + \mathbf{e}_i$ is a demand type in the demand family.* □

Conditions S2 and S3 are useful, but, like any other sufficient conditions, leave some cases uncovered.

Corollary 1. *If a system (Θ, \mathbf{Q}) satisfies Conditions S2 or S3, there exists a transformed system (Φ, \mathbf{Q}_{new}) which satisfies the unit-skeleton condition so that $IP^-(\Theta, \mathbf{Q}) = IP^-(\Phi, \mathbf{Q}_{new}) = 1$.*

Proof. Consider Condition S2. The third sub-condition S2-3 guarantees that the relevant demands are of single units. For any $i \in K_0$, sub-condition S2-1 says that \mathbf{e}_i either exists in the original demand family or can be obtained by applying Proposition 1 to demands differing by \mathbf{e}_i ; similarly, for $i \notin K_0$, sub-condition S2-2 guarantees that \mathbf{e}_i can be obtained by applying Proposition 1 to demands differing by \mathbf{e}_i .

For any i , based on Condition S3, there always exist two demands whose difference is \mathbf{e}_i , which generate \mathbf{e}_i by applying Proposition 1. □

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