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Mingming LENG

Department of Computing and Decision Sciences, Faculty of Business, Lingnan University

Mahmut PARLAR

McMaster University

Dengfeng ZHANG

PingAn Bank, No. 5047 East Shen Nan Road, Shenzhen

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Cooperative Game Analysis of Retail Space-Exchange Problems[1]

Mingming Leng[2, 3], Mahmut Parlar[4], Dengfeng Zhang[5]

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[2]**Corresponding author**: Department of Computing and Decision Sciences, Faculty of Business, Lingnan University, Hong Kong. (Telephone number: +852 2616-8104; Fax number: +852 2892-2442; Email address: mmleng@ln.edu.hk)

[3]Research supported by the General Research Fund of the Hong Kong Research Grants Council under Research Project No. LU340810.

[4]DeGroote School of Business, McMaster University, Hamilton, Ontario, L8S 4M4, Canada.

[5]PingAn Bank, No. 5047 East Shen Nan Road, Shenzhen, Guangdong, 518001, China

Abstract

We analyze retail space-exchange problems where two or more retailers exchange their excess retail spaces to improve the utilization of their space resource. We first investigate the two-retailer space exchange problem. In order to entice both retailers with different bargaining powers to exchange their spaces, we use the generalized Nash bargaining scheme to allocate the total profit surplus between the two retailers. Next, we consider the space-exchange problem involving three or more retailers, and construct a cooperative game in characteristic function form. We show that the game is essential and superadditive, and also prove that the core is non-empty. Moreover, in order to find a unique allocation scheme that ensures the stability of the grand coalition, we propose a new approach to compute a weighted Shapley value that satisfies the core conditions and also reflects retailers' bargaining powers. Our analysis indicates that the space exchange by more retailers can result in a higher system-wide profit surplus and thus a higher allocation to each retailer under a fair scheme.

Key words: Retail space-exchange, bargaining power, generalized Nash bargaining scheme, weighted Shapley value.

1 Introduction

A recent innovation in retailing is the establishment of partnerships by exchanging excess retail space between two (or more) retailers which sell different products. An example of this retail space-exchange strategy is the successful partnership between the British retailers Waitrose (selling food products) and Boots (selling healthcare products). These retailers have established strategic partnership to stock 'selective product ranges' in each others' stores in order to utilize their spaces and increase accessibility to consumers; see Stych [37]. Under this space-exchange strategy, Waitrose can sell its food products in Boots's stores while Waitrose sells Boots's healthcare products in its stores. Encouraged with the success of their space-exchange strategy with Boots, Waitrose has also built a partnership with Welcome Break, which is a British motorway service station operator, opening shops in some stores of the latter. As another successful example of such a space-exchange strategy, in February 2009 Tim Hortons (a favourite doughnut store in Canada) cooperated with Cold Stone Creamery (an U.S.-based chain stores of ice cream) to operate their "co-branded" stores; refer to Draper [17]. In July 2009, Bloomberg BusinessWeek [7] released the news that the two firms had successfully co-branded nearly 50 restaurant locations in the U.S. and Canada within five months, and would continue to greatly expand the partnership. The report in [7] shows the success of the space exchange between Tim Hortons and Cold Stone Creamery.

The success of the space-exchange strategy as implemented by the partnerships formed by Waitrose and Boots, and Tim Hortons and Cold Stone Creamery naturally depends on, (i) whether or not the system-wide profit can be improved, and (ii) whether or not each retailer benefits from this strategy. More specifically, if the two retailers cannot achieve more profit jointly, then this strategy would fail because one or both retailers may be unwilling to cooperate. However, even if the total profit of both retailers increases, then one retailer may still be worse off and thus lose the incentive to cooperate with the other retailer. For this scenario, the retailer who is better off may have to make a transfer payment to compensate for the other retailer's loss. To examine this issue, it is important to consider the fair allocation of the system-wide profit surplus generated by the space exchange. We use the generalized Nash bargaining scheme to analyze the problem.

Since more than two retailers can also cooperate to exchange excess retail spaces, (e.g., Waitrose, Boots and Welcome Break), one should investigate the space exchange problem with three or more retailers. For this problem, it is also important to examine whether or not the system-wide profit can be increased, and thus it is necessary to consider the problem of fair allocation among three or more retailers. For this multiple-retailer problem, we use the weighted Shapley value to determine the fair allocation. We develop a new approach to compute the weighted Shapley value that must be in the core and thus assure the stability of the grand coalition.

The space-exchange strategy results in indirect sales known as the "cross-category effect" for both retailers which should be taken into account when analyzing the retail space-related problems described above. In one of the earliest works to investigate the cross-category effect, Corstjens and Doyle [13] use a space-dependent demand function which can be used to estimate the demand for each retailer. When cross-category effect is not considered, a simpler type of direct sales demand function is obtained. We present a generalized demand model which includes the Corstjens and Doyle's model and the direct sales model as special cases. Our more general model is used to estimate each retailer's sales at his own space and sales at other retailers' spaces.

The remainder of this paper is organized as follows: In Section 2 we review representative relevant publications, and summarize our paper's major contributions to the literature. In Sections 3 and 4 we investigate the space-exchange problem with two retailers, and that with three or more retailers, respectively. In Section 5 we conclude with a summary of major managerial insights that we draw from our analysis.

2 Literature Review

In this section, we briefly review the space-related publications, which are classified into two categories: (i) the space allocation decision for a single retailer, and (ii) the space allocation decisions in the multi-firm settings. Since our paper is concerned with a space decision problem, we only consider representative publications with a focus on the space decision. This also distinguishes our review from a number of review sections in other papers, which present the surveys on various space-related problems without emphasizing the space decisions. Our review is then used to show the originality and importance of our paper.

2.1 The Space Allocation Decision for a Single Retailer

In a seminal paper which is relevant to our discussion, Lee [26] considered a retailer's optimal space allocation decision that maximizes the retailer's profit, assuming that the unit sales for a product are increasing in the shelf space allocated to the product at a decreasing rate. For a review of other early papers published before the middle of the 1970s, see Curhan [14] who shows that the cross-category (or, cross-elasticity) effect—i.e., the impact of sales in a category on the sales in another category—in the retail store was not considered in early publications.

Corstjens and Doyle [13] constructed a multiplicative demand model that involves the cross-category effect between two or among three or more products. Using their demand model, the authors developed a profit function for a retailer, who maximizes his profit to find optimal space allocation decisions. Corstjens and Doyle [13] used a case to estimate the parameters in the retailer's profit function, solved it to find optimal results, and discussed the multiplicative demand model. The demand model in [13] has been widely applied to investigate a variety of space allocation problems. Bookbinder and Zarour [8] extended the Corstjens and Doyle's model [13] by incorporating the concept of direct product profit (DPP), which is calculated as a retailer's gross margin plus total allowance minus total direct product costs. They constructed a constrained maximization problem, estimated parameter values for the problem with two products, and solved the resulting numerical problem to find optimal space allocation decisions.

Borin, Farris, and Freeland [9] also investigated a space allocation problem with the cross-category effect, but constructed a demand model that differs from the Corstjens and Doyle's model [13]. Specifically, the demand model in [9] consists of two components in a multiplicative form, which include (i) each stock-keeping unit's (SKU) proportion of the available demand for non-stocked SKUs, which is described by an attraction model, and (ii) the total demand for non-stocked SKUs. Borin, Farris, and Freeland [9] developed a constrained maximization problem to determine the product selection and find the optimal spaces allocated to those selected products. Irion et al. [24] incorporated the product-specific facings (allocated to each product) into the Corstjens and Doyle's model [13]. The authors used a piecewise linearization technique to reformulate their complicated nonlinear space-allocation model into a linear mixed integer programming problem, which can be solved to obtain near-optimal solutions for large scale optimization problems.

The above publications were only concerned with the space allocation decision for a retailer. We now review representative publications where a retailer makes joint decisions on space allocation and other attributes such as assortment and inventory. As an early publication, Anderson and Amato [1] considered a retailer's joint decisions on the brand selection and the space allocation to each selected brand. A profit function for the retailer was constructed to involve the demand of switching-preference buyers who are allowed to switch from their preferred brands to others. Anderson and Amato [1] derived a necessary condition for the optimal solutions, developed an algorithm to search for the optimal solutions, and discussed the implications of their model. In [20] Hansen and Heinsbroek proposed an algorithm to find a retailer's joint optimal decisions on the product assortment and the space allocation among a given set of products. Different from the model in [1], Hansen and Heinsbroek's model involves the space elasticity of the sales and two constraints on the shelf space allocated to each product.

Baron, Berman, and Perry [5] analyzed the joint shelf space allocation and inventory decisions for a retailer who sells multiple items to satisfy the space- and inventory-dependent demand in the multiplicative function form as in [13]. The authors used the level-crossing theory to analyze their problem, and considered numerical examples with two products to examine the impact of space and inventory on the demand. In [21], assuming a multiplicative demand function for each product on a shelf as in [13], Hariga, Al-Ahmari, and Mohamed built a profit function model for a retailer who simultaneously determines the product assortment, inventory replenishment, display area and shelf space allocation decisions. The authors used the LINGO software to solve the complex mixed integer nonlinear problem.

2.2 The Space Allocation Decisions in the Multi-Firm Settings

In recent years, a few publications have appeared to address the space allocation problems involving multiple firms rather than a single retailer. Kurtuluş and Toktay [25] investigate a space allocation problem for a two-echelon supply chain consisting of a retailer and two competing manufacturers whose products are of different brands but belong to a single category. Each manufacturer makes its wholesale pricing decision, and the retailer determines the shelf space allocated to the category. Assuming a linear, retail price-dependent demand function and a convex space cost for the retailer, the authors obtained the retail prices for two products by using two category management mechanisms—i.e., (i) retailer category management (RCM) where the retailer make the pricing decisions and (ii) category captainship (CC) where a manufacturer acts as the decision maker. Kurtuluş and Toktay [25] showed that the category shelf space under CC may be higher than under RCM, and the shelf space allocated to the category is increasing in the retailer's share of the supply chain-wide profit.

Martínez-de-Albéniz and Roels [15] investigated a retail-space allocation problem for a two-level supply chain where a retailer decides to allocate its shelf space to multiple competing suppliers' products belonging to the same category. The demand for each product was characterized by using the Corstjens and Doyle's model [13]. The multiple suppliers first make their wholesale pricing/attraction (the retailer's maximum profit from each product) decisions in Nash equilibrium, and the retailer then makes its optimal space allocation decision. Martínez-de-Albéniz and Roels [15] analyzed their problem with both endogenous and exogenous retail prices, and examined the loss of efficiency resulting from shelf space competition among the suppliers by using numerical examples with two products.

Leng, Parlar, and Zhang [27] investigated the pricing and space allocation decisions for two (non-cooperating) retailers whose stores are located at two end points of a linear city. They used the Hotelling model to analyze consumers' choices for their shopping stores and to estimate the demand functions for two retailers, derived the two retailers' optimal prices given the space allocation in each store, and then obtained the two retailer's Nash equilibrium space allocation decisions. Leng, Parlar, and Zhang [27] showed that the two retailers should adopt the space-exchange strategy, if their stores are large enough to serve at least one-half of their consumers. In addition, the authors found that the space exchange induces the two retailers to increase their prices.

2.3 General Remarks

Our review shows that majority of extant publications have used or extended the Corstjens and Doyle's demand model [13] to analyze the space allocation problem. Moreover, most of the relevant publications are concerned with a single retailer's space decision. Among a few recent publications that address space allocation problems with two or more firms, only one publication (i.e., [27]) investigated the competition between two retailers, whereas other publications considered two-echelon supply chains involving multiple suppliers/manufacturers and a retailer. Our paper's major contributions to the literature are summarized as follows:

1. We generalize the Corstjens and Doyle's model [13] to estimate the demand for each product category in space exchange problems. Our generalized model includes the Corstjens and Doyle's model and the direct sales model as special cases.
2. Our paper addresses the space exchange problem between two retailers, which is a new research topic in the marketing and operations management areas. We draw a number of managerial insights that are different from Leng, Parlar, and Zhang [27] who used non-

cooperative game theory to analyze the space exchange-related problem. Our managerial insights are expected to help practitioners benefit from the space-exchange strategy. For a summary of our six major insights, see Section 5.

3. Our paper significantly differs from [27] because of the following four facts. First, we use the space-dependent demand functions in a multiplicative form (as in Corstjens and Doyle [13]), whereas Leng, Parlar, and Zhang [27] applied the Hotelling model to derive the price-dependent demand functions. Secondly, we focus on the fair allocation of profit surplus resulting from the space exchange, which assures the retailers' incentives to cooperate for such a strategy. This was ignored in [27], where the authors assumed that two retailers are willing to cooperate. Thirdly, in our paper, both the space exchange problem with two retailers and that with three or more retailers are analyzed, whereas in [27] only the two-retailer problem was investigated. Fourthly, we examine a more realistic case in which the retailers under the space exchange strategy may have different bargaining powers, whereas Leng, Parlar, and Zhang [27] assumed that two retailers have equal bargaining powers.

   We also note that, to investigate the space-exchange problems, we use cooperative game theory but Leng, Parlar, and Zhang [27] apply non-cooperative game theory. This is so mainly because our paper is focused on retailers' *incentives* for exchanging their spaces, whereas Leng, Parlar, and Zhang [27] emphasize the *impact* of space exchange on retailers' pricing decisions under the assumption that the retailers are willing to cooperate.
4. We provided a new approach for the computation of weighted Shapley value, which is an important concept in cooperative game theory. Our approach should be of help to other researchers who consider the allocation of profit surplus or cost savings among multiple players with different bargaining powers.

3 The Space-Exchange Problems with Two Retailers

In this section we consider a two-player space-exchange problem where two retailers exchange their retail spaces to increase sales. As indicated by the practice of Waitrose and Boots and also by that of Tim Hortons and Cold Stone Creamery, the retail space-exchange strategy applies only when the cooperating retailers' products belong to unsubstitutable categories, e.g., Waitrose's food vs. Boots's healthcare products; and, Tim Hortons's doughnuts vs. Cold Stone Creamery's ice cream. Thus, we can reasonably assume that the products in categories $i = 1, 2$ sold by retailer $i = 1, 2$, are not substitutable.

The total retail space that is owned by retailer i is denoted by $S_i > 0$ for $i = 1, 2$. To implement the space-exchange strategy, retailer i—who sells the products in category i—decides to allocate the retail space $S_{ij} \in [0, S_i]$ ($j = 1, 2,\ j \neq i$) to retailer j who can then sell the products of category j using the space S_{ij} at the site of retailer i. As a result of the space exchange, retailer i sells the products of category i in the remaining space $S_{ii} \equiv S_i - S_{ij}$ at his own store as the "host retailer," and also in the new retail space S_{ji} at retailer j as the "guest retailer." Even though a retailer sells his products at both stores, the realized sales in the "host" and "guest" spaces could be different because of the following factors: (i) Consumers who intend to buy the products in category i would more likely visit the store owned by retailer i, since the store brand certainly impacts their purchasing decisions, and (ii) the "host" and "guest" stores are located at different neighborhoods where consumers may have different purchasing powers and behaviors. In this paper, we use the term *store effect* to describe the impacts of the above two factors, and develop different functions to model the realized sales of each retailer at his "host" and "guest" retail spaces.

Next, we first analyze a retail space-exchange game where two retailers determine their "host" retail spaces with no communication. We solve the "simultaneous-move" game to find two retailers' decisions in Nash equilibrium, and then compute the retailer's *excess* retail space as the total owned space S_i $(i = 1, 2)$ minus the equilibrium host space. The excess space is what the retailer occupies but does not need to improve his profitability, and both retailers exchange their excess spaces to effectively use up the resource.

We begin our analysis by developing the profit functions of the two retailers.

3.1 Profit Functions of Two Retailers

After the retailers exchange their excess spaces, retailer $i = 1, 2$ realizes sales generated at both host and guest spaces. Let $D_{ii}(S_i - S_{ij})$ denote retailer i's *realized* sales at his host space $S_i - S_{ij}$ after retailer i allocates the space S_{ij} to retailer j $(j = 1, 2,\ j \neq i)$; and let $D_{ij}(S_{ji})$ denote retailer i's *realized* sales at his guest space S_{ji} that is allocated by retailer j. Since $S_{ii} \equiv S_i - S_{ij}$, we can simply write retailer i's sales at the site of retailer j as $D_{ij}(S_{ji})$, for $i, j = 1, 2$. To simplify our analysis and obtain useful managerial insights, we define each retailer's realized sales at either the host or guest space as the minimum of, (i) that retailer's available stock, and, (ii) the customers' total demand during one year. If we do not use the above definition, then we have to consider each retailer's stocking decision to determine his realized sales. Note that we will use a multiplicative demand model, apply non-cooperative game theory to find space allocation decisions, and use cooperative game theory to allocate the profit surplus between two or among three or more retailers. Thus, if, for space-exchange problems, each retailer makes joint stocking and space allocation decisions, then our models will be intractably complicated and we cannot find any insightful result. Therefore, we adopt the above definition to simplify our analysis, similar to, e.g., Wang and Gerchak [40] who analyzed a space-related problem assuming that there is always sufficient inventory to satisfy deterministic demand.

For our space-exchange problem, retailer i's realized sales consist of *direct* and *indirect* sale components. The *direct* realized sales of retailer i are the sales generated as a result of his effort and *independent* of the space exchange decisions. This means that even if the two retailers do not exchange their excess spaces, they can realize the same direct sales as those in the space-exchange case (if total operating space of each retailer is the same). In addition to the realized direct sales, we also consider the *indirect* sales that are generated by exchanging retail space.

Note that the space-exchange strategy results in two categories of products available for sale at the stores of both retailers. A consumer who intends to buy a category 1 product (called *lead category*) may also purchase a category 2 product, and vice versa. (For details regarding the lead category concept, see, e.g., Chen et al. [12] who used the term "potential sales" rather than "indirect sales".) Since exchanging retail spaces may entice the consumers with lead category i ($i = 1, 2$) to buy products in category j ($j = 1, 2$, and $j \neq i$), the space-exchange strategy results in *indirect sales* for both retailers. Such an impact of the space exchange is called the *cross-category effect.*

The cross-category effect in retail space-related problems has been investigated by a number of researchers in the marketing and operations management fields; see, e.g., Chen et al. [12], Corstjens and Doyle [13], Hruschka et al. [23], Lim et al. [28], Niraj et al. [32]. A few extant publications [e.g., Chen et al. [12]] developed some linear functions to calculate the indirect sales. To estimate the cross-category effect in a nonlinear way, most relevant publications [e.g., Lim et al. [28]] adopted the Corstjens and Doyle's space-dependent sales (demand) model in a polynomial form [13]. For such space-related problems, the Corstjens and Doyle's model is commonly used to calculate retailer i's sales in the space S_{ji} at the site of retailer j as,

$$D_{ij}(S_{ji}) = \alpha_{ij} S_{ji}^{\beta_i^j} (S_j - S_{ji})^{\delta_{i\ell}^j}, \quad \text{for } i, j, \ell = 1, 2, \quad \ell \neq i. \tag{1}$$

In this model, $\alpha_{ij} > 0$ denotes the scale parameter; S_{ji} and $S_j - S_{ji}$ are two retailers' shelf spaces at the site of retailer j (as defined previously); $0 < \beta_i^j < 1$ represents the *direct* space elasticity that affects the *direct sales* of retailer i at the site of retailer j; and $0 < \delta_{i\ell}^j < 1$ is the *cross* space elasticity between categories i and ℓ (because $i, \ell = 1, 2$ and $i \neq \ell$), which impacts the *indirect sales* of retailer i at the site of retailer j.

The Corstjens and Doyle's model (1) has been widely used to analyze space-related problems. But, we cannot apply it to calculate two retailers' sales in our space-exchange problem, because of the following reasons:

1. When retailer i occupies the total available space S_j at the site of retailer j, i.e., $S_{ji} = S_j$, the Corstjens and Doyle's model in (1) implies that retailer i's sales is zero. This is not realistic because retailer i utilizes S_j to operate and thus realizes a non-zero direct sales. In fact, if the cross-category effect is not considered, then retailer i's (direct) sales function should be simply written as
$$D_{ij}(S_j) = \alpha_{ij} S_j^{\beta_i^j}, \tag{2}$$
which is a broadly-used space- or inventory-dependent demand function; see, e.g., Baker and Urban [3], Bar-Lev et al. [4], Brueckner [10], Wang and Gerchak [40]. For a detailed discussion regarding the advantages of the model in (2), see Baker and Urban [3]. One may note that, when $S_{ji} = S_j$, the Corstjens and Doyle's model (1) cannot be reduced to the direct sales function in (2).
2. Consider the case where retailer i does not use the total available space S_j, i.e., $S_{ji} < S_j$. In this case, if $S_j - S_{ji} < 1$, then, because $\delta_{i\ell}^j > 0$, retailer i's sales $D_{ij}(S_{ji})$ is smaller

than $\alpha_{ij} S_{ji}^{\beta_i^j}$, which represents the direct sales achieved by retailer i's own effort. That is, using the Corstjens and Doyle's model in (1), we find that, if $S_j - S_{ji} < 1$, then retailer i's sales may be reduced as a consequence of the cross-category effect. Such a result (for the case that $S_j - S_{ji} < 1$) is inconsistent with other relevant publications [e.g., Chen et al. [12], Lim et al. [28], Niraj et al. [32]], in which researchers found that the cross-category effect should have a positive impact on the performance of retailers whose products are not substitutable.

Taking the above into account, we generalize the Corstjens and Doyle's model to describe the sales that retailer i ($i = 1, 2$) realizes in the space S_{ji} at the site of retailer j ($j = 1, 2$). Our model is given as,

$$D_{ij}(S_{ji}) = S_{ji}^{\beta_i^j}[\alpha_{ij} + \hat{\alpha}_{ij}(S_j - S_{ji})^{\delta_{i\ell}^j}], \quad \text{for } i, j, \ell = 1, 2, \quad \ell \neq i, \tag{3}$$

where $\alpha_{ij}, \hat{\alpha}_{ij} \geq 0$ can be regarded as the scale parameters for the direct and indirect sales, respectively. It is easy to see that when $\alpha_{ij} = 0$, our generalized model and the Corstjens and Doyle's model in (1) have the same structure; and when $\hat{\alpha}_{ij} = 0$, our model can be reduced to the commonly-used direct sales model in (2). Actually, the model in (3) is suitable to the analysis of space-exchange problems because of the following reason: Under the space-exchange strategy, two or more retailers exchange their excess spaces each other. As a result, there are two or more product categories for sale at the store of each retailer in the space exchange alliance. Recall from our literature review in Section 2.1 that Corstjens and Doyle [13] developed the model in (1) to characterize the demands for $n \geq 2$ products. The Corstjens and Doyle's model has been used to investigate various space-related problems in a number of publications, among which some publications (e.g., [5], [8], [15], etc.) applied the Corstjens and Doyle's model to determine the space allocation between two products. Therefore, the above shows that the Corstjens and Doyle's model in (1) is proper to the space allocation problems with two or more products, and our generalized model in (3) are thus suitable to the space exchange problems in which each retailer decides to allocate the space at his own store between his product category and the other retailer's product category or among his category and the categories of other two or more retailers.

We let $m_i > 0$ denote retailer i's profit margin measured by the sales profit per unit space, e.g., dollars per square foot. Some marketing researchers [e.g., Anderson et al. [2] and Blattberg and Neslin [6]] have shown that the profit margins of different products in the same category are typically identical and setting a uniform margin has been a common pricing rule for retailers. In our paper, the products that each retailer sells belong to a single category; thus, it is reasonable to assume that each retailer applies an identical profit margin to his products. This assumption has been used in marketing- and operations management-related publications; see, for example, Cachon and Kok [11], Dong et al. [16]. The operating cost at retailer i's store is h_i dollars per unit retail space per year. Thus, if two retailers do *not* exchange their excess spaces, their profit functions are simply

$$\pi_i(S_i) = m_i D_{ii}(S_i) - h_i S_i, \quad \text{for } i = 1, 2. \tag{4}$$

Now consider the two retailers' sales profits after they *exchange* their retail spaces $S_{ij} \in [0, S_i]$, for $i, j = 1, 2$, $i \neq j$. Retailer i incurs the operating cost of $h_i(S_{ii})$ in his host space, and $h_j(S_j - S_{jj})$ in his guest space. We write the two retailers' post-exchange net profit functions as

$$\pi_i(S_{ii}; S_{jj}) = m_i[D_{ii}(S_{ii}) + D_{ij}(S_j - S_{jj})] - h_i S_{ii} - h_j(S_j - S_{jj}), \text{ for } i, j = 1, 2 \text{ and } i \neq j, \tag{5}$$

where $D_{ii}(S_{ii})$ and $D_{ij}(S_j - S_{jj})$ are retailer i's realized sales in his host space S_{ii} and those in the guest space $S_j - S_{jj}$ (that retailer j gives to retailer i), respectively.

3.2 The Space-Exchange Analysis with Two Retailers

We now consider the space-exchange problem with two retailers who determine their spaces that maximize their profits in a simultaneous-move setting, and exchange their excess spaces. We then use cooperative game theory to allocate the profit surplus between the retailers.

3.2.1 Computation of Excess Spaces

We solve a space-exchange game where two retailers' profits are given as in (5). Note that, to willingly implement the space-exchange strategy, two retailers should benefit from space exchange by achieving a higher profit (or, a positive profit surplus). We next calculate the profit surpluses that two retailers obtain per year after exchanging their excess spaces. Let $\Delta\pi_i$ denote retailer i's profit surplus generated by the space exchange. We use equations (4) and (5) to find the profit surpluses as $\Delta\pi_i$ $(i = 1, 2)$,

$$\Delta\pi_i = \pi_i(S_{ii}; S_{jj}) - \pi_i(S_i) = m_i[D_{ii}(S_{ii}) - D_{ii}(S_i) + D_{ij}(S_j - S_{jj})] + h_i(S_i - S_{ii}) - h_j(S_j - S_{jj}), \tag{6}$$

for $j = 1, 2$ and $j \neq i$.

If $\Delta\pi_1 + \Delta\pi_2 < 0$, then the two retailers cannot concurrently benefit from the space exchange, and they should not be willing to exchange their excess spaces. When $\Delta\pi_1 + \Delta\pi_2 \geq 0$, one of the two retailers may still experience a profit loss, i.e., either $\Delta\pi_1$ or $\Delta\pi_2$ may be negative. For this case, the retailer with a positive profit surplus can allocate a part of his surplus to the other retailer, in order to assure that both retailers are better off from space exchange. Hence, we can conclude that the condition for the successful space exchange is $\Delta\pi_1 + \Delta\pi_2 \geq 0$.

Using the above we find that, to make a space allocation decision, each retailer should consider the constraint that $\Delta\pi_1 + \Delta\pi_2 \geq 0$. That is, in the simultaneous-move game, retailer i's $(i = 1, 2)$ constrained maximization problem is written as follows:

$$\max \pi_i(S_{ii}; S_{jj}), \quad \text{s.t., } \Delta\pi_1 + \Delta\pi_2 \geq 0 \text{ and } 0 \leq S_{ii} \leq S_i. \tag{7}$$

Retailer i determines his Nash equilibrium-characterized host space S_{ii}^N, and compute excess spaces (that are then allocated to the other) as this retailer's total own space S_i minus the host space S_{ii}^N.

Theorem 1 The host space S_{ii}^N $(i = 1, 2)$ in Nash equilibrium is obtained as

$$S_{ii}^N = \min(\max(\hat{S}_{ii}, z_i), S_i) = \begin{cases} z_i, & \text{if } \hat{S}_{ii} \leq z_i, \\ \hat{S}_{ii}, & \text{if } z_i < \hat{S}_{ii} \leq S_i, \\ S_i, & \text{if } \hat{S}_{ii} \leq S_i, \end{cases} \tag{8}$$

where $z_i \equiv \{S_{ii} : D_{ii}(S_{ii}) = D_{ii}(S_i) \text{ and } S_{ii} \neq S_i\}$ is a unique value in the range $(0, S_i)$; and $\hat{S}_{ii}$ uniquely satisfies

$$\alpha_{ii}\beta_i^i S_{ii}^{\beta_i^i - 1} + \hat{\alpha}_{ii} S_{ii}^{\beta_i^i - 1}(S_i - S_{ii})^{\delta_{i\ell}^i - 1}[\beta_i^i(S_i - S_{ii}) - \hat{\alpha}_{ii}\delta_{i\ell}^i S_{ii}] = h_i/m_i. \tag{9}$$

Proof. For a proof of this theorem and the proofs for all subsequent theorems, see online Appendix A. ■

It follows from Theorem 1 that the optimal excess space that retailer i $(i = 1, 2)$ allocates to retailer j $(j = 1, 2,\ j \neq i)$ is calculated as,

$$S_{ij}^N = S_i - S_{ii}^N = [S_i - \max(\hat{S}_{ii}, z_i)]^+, \quad \text{for } i, j = 1, 2 \text{ and } i \neq j, \tag{10}$$

where $x^+ = \max(0, x)$ for any x, and $\hat{S}_{ii}$ is dependent on the ratio of unit operating cost h_i to profit margin m_i. From (9), we find that the retailer with a higher value of h_i/m_i should secure less host space and allocate more to the other retailer.

3.2.2 Allocation of System-Wide Profit Surplus

When two retailers exchange their optimal excess spaces as given in (10), they will jointly benefit from space exchange by achieving the total profit surplus $\Delta\pi_1 + \Delta\pi_2 \geq 0$. As discussed in Section 3.2.1, even when $\Delta\pi_1 + \Delta\pi_2 \geq 0$, one of the two retailers may still experience a profit loss, i.e., either $\Delta\pi_1$ or $\Delta\pi_2$ may be negative. For example, if the unit operating cost h_2 at retailer 2's store is sufficiently high, then retailer 1's profit surplus $\Delta\pi_1$ could become negative but retailer 2's profit surplus $\Delta\pi_2$ could be positive; similarly, if h_1 is very high, then retailer 2 may have negative profit surplus but retailer 1 may achieve positive surplus.

Thus, when $\Delta\pi_1 + \Delta\pi_2 \geq 0$, we need to consider the following three possible cases:

Case 1: $\Delta\pi_1 \geq 0$ and $\Delta\pi_2 \geq 0$. In this case, neither retailer is worse off after the excess space exchange; but they may bargain on allocating the system-wide profit surplus, because of the following reason: When a retailer's profit surplus is very small (e.g., retailer 1's profit surplus $\Delta\pi_1 = 0$) and the other retailer's profit surplus is large (e.g., retailer 2's profit surplus $\Delta\pi_2 > 0$), we find that if the latter retailer does not share a part of his large surplus to the former retailer, then the former may have no incentive to cooperate with the latter. It behooves us to consider the question of how to fairly allocate the profit surplus between two retailers so as to induce coalition stability. Under a fair allocation scheme, both retailers would be willing to exchange their excess space and they would be both better off than leaving the two-player coalition. Moreover, the fair allocation of total profit surplus between these two retailers should depend on their relative bargaining powers. Later we will use the concept of generalized Nash bargaining scheme to analyze such an allocation problem.

Case 2: $\Delta\pi_1 > 0$ **and** $\Delta\pi_2 < 0$**.** In this case, retailer 1 benefits from the space exchange and gains a positive profit surplus; but, retailer 2 is worse off than without space exchange. In order to attract retailer 2 to cooperate and make the two-player coalition stable, retailer 1 would be willing to share his profit surplus $\Delta\pi_1$ with retailer 2. Similar to Case 1, we also need to consider the fair allocation between these two retailers for this case.

Case 3: $\Delta\pi_1 < 0$ **and** $\Delta\pi_2 > 0$**.** This case is similar to Case 2, but retailer 1 is worse off whereas retailer 2 achieves a positive surplus $\Delta\pi_2$.

As we discussed above, when two retailers make their decisions as given in Theorem 1, we should analyze the problem of fairly allocating the system-wide profit surplus $\Delta\pi_1 + \Delta\pi_2$ between retailers 1 and 2. We start with the following discussion on the fairness criteria for the allocation of the total profit surplus. We learn from Theorem 1 that two retailers are not worse off than without exchanging their excess spaces; this means that the space exchange decisions in Nash equilibrium are "acceptable" to both retailers. However, a retailer who only "accepts" an allocation method may still feel that he is treated unfairly and thus leave the two-player coalition, if that retailer's secured allocation cannot reflect his bargaining power for the allocation. Therefore, as discussed in many allocation-related publications (e.g., Hamlen et al. [19]), a fair allocation scheme must be acceptable to both retailers and also be compatible with the retailers' relative bargaining powers. We present below a definition to explicitly explain the allocation fairness used in our paper.

Definition 1 A scheme of allocating total profit surplus between or among all retailers in a coalition is fair, if and only if the allocation scheme reflects these retailers' relative bargaining powers. ♦

In our retail space-exchange problem, retailers 1 and 2 *fully* divide the surplus $\Delta\pi_1 + \Delta\pi_2$ in such a manner that each retailer gains a portion of $\Delta\pi_1 + \Delta\pi_2$ that reflects the retailer's bargaining power. To facilitate our analysis, we use the notation y_i to denote the surplus allocated to retailer i, for $i = 1, 2$; that is, we consider how to fairly determine the values of y_1 and y_2 such that $y_1 + y_2 = \Delta\pi_1 + \Delta\pi_2 \geq 0$. For our space-exchange problem, we assume that, to allocate total profit surplus $\Delta\pi_1 + \Delta\pi_2$, retailer 1's and retailer 2's bargaining powers are denoted by γ_1 and γ_2, respectively.

To fairly allocate the surplus $\Delta\pi_1 + \Delta\pi_2$ between these two retailers, we use generalized Nash bargaining (GNB) scheme (see Nash [31] and Roth [33]), which is a unique solution satisfying the following maximization problem:

$$\max_{y_1 \geq 0, y_2 \geq 0} \; y_1^{\gamma_1} y_2^{\gamma_2}, \quad \text{s.t. } (y_1, y_2) \in \mathcal{P}, \tag{11}$$

where y_i and y_i^0 correspond to retailer i's allocated surplus, and security level, respectively, $i = 1, 2$; and $\mathcal{P}$ denotes the set of Pareto optimal solutions. Note that, in our problem, $(y_1^0, y_2^0) = (0, 0)$. Moreover, because two retailers *fully* share the system-wide profit surplus $\Delta\pi_1 + \Delta\pi_2$, the set of Pareto optimal solutions is $\mathcal{P} = \{(y_1, y_2) : y_1 + y_2 = \Delta\pi_1 + \Delta\pi_2\}$. The concept of GNB scheme has been used to analyze supply chain-related problems, see, e.g., Nagarajan and Bassok [30].

Theorem 2 When two retailers keep their host spaces as obtained in Theorem 1 and exchange their excess spaces in (10), the GNB scheme suggests that, in order to entice both retailers to join the two-player coalition, they should share the system-wide profit surplus $\Delta\pi_1 + \Delta\pi_2$ as follows:

$$y_i^* = (\Delta\pi_1 + \Delta\pi_2)\frac{\gamma_i}{\gamma_1 + \gamma_2}, \quad \text{for } i = 1, 2. \quad \blacksquare \tag{12}$$

As Theorem 2 indicates, the allocation suggested by the GNB scheme depends on two retailers' relative bargaining powers, i.e., $\gamma_i/(\gamma_1 + \gamma_2)$, for $i = 1, 2$. Note that retailers 1 and 2 respectively realize their own profit surpluses $\Delta\pi_1$ and $\Delta\pi_2$ before the allocation of $\Delta\pi_1 + \Delta\pi_2$, and obtain the GNB allocations y_1^* and y_2^* that are given in Theorem 2. Since $\Delta\pi_i$ may not be equal to y_i^*, for $i = 1, 2$, we need to calculate the side-payment transferred between these two retailers under the GNB allocation scheme.

Since retailer $i = 1, 2$ realizes the profit surplus $\Delta\pi_i$ and obtains the GNB allocation y_i^*, we can compute the non-negative side-payment η_{ij} that this retailer transfers to retailer j ($j = 1, 2$, $j \neq i$) as follows:

$$\eta_{ij} = (\Delta\pi_i - y_i^*)^+ = \left(\frac{\gamma_j \cdot \Delta\pi_i - \gamma_i \cdot \Delta\pi_j}{\gamma_i + \gamma_j}\right)^+. \tag{13}$$

Because $\Delta\pi_1 + \Delta\pi_2 = y_1^* + y_2^*$, one of η_{12} and η_{21} must be zero. If $\eta_{12} = \eta_{21} = 0$, then $\Delta\pi_1 = y_1^*$ and $\Delta\pi_2 = y_2^*$, and there is no side-payment between two retailers; if $\eta_{12} > 0$ and $\eta_{21} = 0$, then retailer 1 transfers the side-payment η_{12} to retailer 2; otherwise, retailer 2 transfers the side-payment η_{21} to retailer 1.

It is useful to note from (13) that, when $\Delta\pi_1 > 0$ and $\Delta\pi_2 > 0$, which each retailer makes the payment transfer depends on the comparison between the ratios $\Delta\pi_1/\Delta\pi_2$ and γ_1/γ_2. The ratio $\Delta\pi_1/\Delta\pi_2$ represents the *relative* profit surpluses of two retailers at their own sites; and the ratio γ_1/γ_2 is two retailers' relative bargaining powers. For instance, if, compared with retailer 2, retailer 1's bargaining power is strong but his own profit surplus is small, then retailer 1 should gain more to have an incentive to stay in the two-retailer coalition for the space exchange; thus, for this case, retailer 2 should transfer a side-payment to retailer 1.

3.2.3 A Numerical Example for the Space Exchange Analysis with Two Retailers

We use the parameter values given in Table 1 for our computation.

For the space exchange problem with two retailers, we first use Theorem 1 to compute the Nash equilibrium-based host spaces for two retailers. We find that $\hat{S}_{11} = 67.23 < z_1 = 82.68$

	m_1	m_2	h_1	h_2	S_1	S_2	γ_1	γ_2	δ_{12}^1	δ_{12}^2	
	15	25	4	6	100	60	3	8	0.1	0.2	
α_{11}	$\hat{\alpha}_{11}$	α_{12}	$\hat{\alpha}_{12}$	α_{21}	$\hat{\alpha}_{21}$	α_{22}	$\hat{\alpha}_{22}$	β_1^1	β_1^2	β_2^1	β_2^2
4	0.3	1	0.2	3	0.3	6	0.4	0.5	0.4	0.3	0.4

Table 1: The parameter values in the numerical example for the space exchange analysis with two retailers. Note that profit margins m_i $(i = 1, 2)$, operating costs h_i $(i = 1, 2)$ and two retailers' total available host spaces S_i $(i = 1, 2)$ are measured in "\$/unit", "\$/sq. ft" and "sq. ft.", respectively.

and $z_2 = 45.89 < \hat{S}_{22} = 48.07 < S_2$; thus, according to (8), Retailer 1's and retailer 2's host spaces in Nash equilibrium are obtained as $S_{11}^N = z_1 = 82.68$ sq. ft. and $S_{22}^N = \hat{S}_{22} = 48.07$ sq. ft. and the excess spaces that the two retailers exchange are thus calculated as $S_{12}^N = S_1 - S_{11}^N = 17.32$ sq. ft.—which is allocated by retailer 1 to retailer 2—and $S_{21}^N = S_2 - S_{22}^N = 11.93$ sq. ft.—which is allocated by retailer 2 to retailer 1.

After two retailers exchange their excess spaces, we use (6) to find the profit surpluses of retailers 1 and 2 as $\Delta\pi_1 = \$55.67$ and $\Delta\pi_2 = \$218.01$, which means that retailer 2 gains much more from the space-exchange than retailer 1. In order to entice retailer 1 to exchange his excess space, we should fairly allocate the system-wide profit surplus $\Delta\pi_1 + \Delta\pi_2 = \$273.68$ between the two retailers. We follow Theorem 2 to calculate the GNB allocation as $y_1^* = \$74.64$ and $y_2^* = \$199.04$, which reflect the two retailers' relative bargaining powers. To implement the allocation scheme, we calculate $\eta_{21} = \Delta\pi_2 - y_2^* = \$18.97$, which means that retailer 2 should make the side-payment \$18.97 to retailer 1. As a result, two retailers will have the profit surpluses (y_1^*, y_2^*).

4 The Space-Exchange Problems with Three or More Retailers

We consider the space exchange problems where n retailers $(n \geq 3)$ exchange their excess spaces to efficiently utilize the total space resource. We first construct each retailer's profit function and find space allocation decisions in Nash equilibrium. Then, in order to examine the stability of various coalition structures for the space exchange, we develop a cooperative game model in characteristic-function form, and use the cooperative game concepts (i.e., the core, the weighted Shapley value) to investigate the fair allocation of the system-wide profit surplus among $n \geq 3$ retailers.

4.1 Profit Functions of Retailers in a Space-Exchange Coalition

In the space exchange problem with $n \geq 3$ retailers (denoted by $1, 2, \ldots, n$), each retailer may join any possible coalition where the retailers determine their optimal host spaces and then exchange excess spaces with each other. Since any subset of k $(1 \leq k \leq n)$ retailers may form a k-retailer coalition, there are $\binom{n}{k} = n!/[k!(n-k)!]$ possible k-retailer coalitions which we denote by $C(r;k)$, for $r = 1, 2 \ldots, \binom{n}{k}$ and $1 \le k \le n$. Moreover, if retailer $i = 1, 2, \ldots, n$ joins the rth k-retailer coalition $C(r;k)$, then we denote this retailer's ordered index in this coalition by $I_i(r;k) = 1, \ldots, k$; otherwise, if the k-retailer coalition $C(r;k)$ does not include retailer i, then this retailer's ordered index in this coalition is $I_i(r;k) = 0$. For example, in a 3-retailer game with retailers 1, 2, and 3, there are three possible 2-retailer coalitions that are denoted by $C(1;2) = (1,2)$, $C(2;2) = (1,3)$ and $C(3;2) = (2,3)$. Retailer 1's ordered index in the coalition $C(1;2)$ is $I_1(1;2) = 1$ and his ordered index in the coalition $C(2;2)$ is $I_1(2;2) = 1$; but, retailer 1's ordered index in coalition $C(3;2)$ is $I_1(3;2) = 0$. The ordered indices of retailers 2 and 3 in all 2-retailer coalitions are found in a similar manner.

Without loss of generality, we now consider the rth k-retailer coalition $C(r;k)$ involving retailers $1, 2, \ldots, k$. Similar to Section 3.1, we can compute the realized profit of retailer $i \in C(r;k)$ who allocates his host space $S_{ij}^{(r;k)} \in [0, S_i]$ to retailer $j \in C_{-i}(r;k) \equiv C(r;k) \backslash \{i\}$, and receives the guest space $S_{ji}^{(r;k)} \in [0, S_j]$ from retailer j. More specifically, *before* retailer i cooperates with other retailers for the space exchange, this retailer only operates at his host space S_i, and obtains the net profit of

$$\pi_i(S_i) = m_i D_{ii}(S_i) - h_i S_i = m_i \alpha_{ii} S_i^{\beta_i^i} - h_i S_i, \quad \text{for } i \in C(r;k). \tag{14}$$

Next, we calculate retailer i's profit *after* the retailer exchanges his excess space with the other retailers in the coalition $C(r;k)$. Similar to Section 3.1, we generalize the Corstjens and Doyle's model [13] to develop a sales model for retailer $i \in C(r;k)$ at the site of retailer $j \in C(r;k)$ as,

$$D_{ij}(S_{ji}^{(r;k)}) = (S_{ji}^{(r;k)})^{\beta_i^j} \times \prod\nolimits_{\ell \in C_{-i}(r;k)} [\alpha_{ij}^{\ell} + \hat{\alpha}_{ij}^{\ell} (S_{j\ell}^{(r;k)})^{\delta_{i\ell}^j}], \tag{15}$$

where $0 < \beta_i^j < 1$ denotes the *direct* space elasticity for retailer i at the site of retailer j; $0 < \delta_{i\ell}^j < 1$ denotes the *cross* space elasticity between category i [for retailer $i \in C(r;k)$] and category ℓ [for retailer $\ell \in C_{-i}(r;k)$] at the site of retailer j; $\alpha_{ij}^{\ell}, \hat{\alpha}_{ij}^{\ell} > 0$ respectively mean the scale parameters at the site of retailer j.

When retailer i is a member in the rth k-retailer coalition $C(r;k)$ [that is, $i \in C(r;k)$] and exchanges his excess space with the other retailers, this retailer can realize the profit $\pi_i(\mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$, where

$$\mathbf{S}_{-i}^{(r;k)} \equiv \{S_{ij}^{(r;k)} : j \in C_{-i}(r;k)\} \quad \text{and} \quad \mathbf{S}_{+i}^{(r;k)} \equiv \{S_{ji}^{(r;k)} : j \in C_{-i}(r;k)\},$$

which, respectively, denote the set of retailer i's excess spaces allocated to retailer $j \in C_{-i}(r;k)$ and the set of the excess spaces that retailer i receives from the other $(k-1)$ retailers (i.e., retailers $j \in C_{-i}(r;k)$). In addition, we define total excess space $(S_{-i}^{(r;k)})$ that retailer $i \in C(r;k)$ allocates to other retailers in the rth k-retailer coalition $C(r;k)$, and total excess space $(S_{+i}^{(r;k)})$ that retailer i receives from other retailers, as follows:

$$S_{-i}^{(r;k)} \equiv \sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)}, \quad \text{and} \quad S_{+i}^{(r;k)} \equiv \sum_{j \in C_{-i}(r;k)} S_{ji}^{(r;k)}.$$

Similar to Section 3.1, the profit function of retailer $i \in C(r;k)$ is developed as

$$\pi_i(\mathbf{S}_i^{(r;k)}; \mathbf{S}_{+i}^{(r;k)}) = m_i \sum_{j \in C(r;k)} D_{ij}(S_{ji}^{(r;k)}) - \sum_{j \in C(r;k)} h_j S_{ji}^{(r;k)}, \tag{16}$$

where $\mathbf{S}_i^{(r;k)} = \{S_{ii}\} \cup \mathbf{S}_{-i}^{(r;k)}$. Note that, in (16), when $j = i$, $S_{ii}^{(r;k)} = S_i - S_{-i}^{(r;k)}$, which denotes retailer i's host space that is used for this retailer's own sale; and $D_{ij}(S_{ji}^{(r;k)}) = D_{ii}(S_{ii}^{(r;k)})$, which represents the profit that retailer $i \in C(r;k)$ can achieve by using the host space $S_{ii}^{(r;k)}$ at his own site.

4.2 The Space-Exchange Analysis with $n \geq 3$ Retailers

We now consider the n-retailer space exchange problem where all retailers in each coalition make their space allocation decisions in Nash equilibrium. More specifically, in the rth k-retailer coalition $C(r;k)$ (for $r = 1, 2 \ldots, \binom{n}{k}$ and $1 \leq k \leq n$), all retailers maximize their profits in (16) in a simultaneous-move setting to make space allocation decisions (i.e., $S_{ij}^{(r;k)N}$, $\forall i, j \in C(r;k)$). Similar to Section 3.2, retailer $i \in C(r;k)$ need to solve the following constrained maximization problem:

$$\max_{S_{ij}^{(r;k)}, j \in C(r;k)} \pi_i(\mathbf{S}_i^{(r;k)}; \mathbf{S}_{+i}^{(r;k)}), \quad \text{s.t.} \sum_{j \in C(r;k)} \Delta\pi_j \geq 0 \text{ and } \sum_{j \in C(r;k)} S_{ij}^{(r;k)} = S_i, \tag{17}$$

where $\pi_i(\mathbf{S}_i^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$ is given as in (16); the first constraint is considered to assure that all retailers in the coalition $C(r;k)$ are willing to exchange their spaces; and the second constraint assures that retailer i fully allocate the total space at his own site among all retailers (including retailer i himself) in the coalition $C(r;k)$. Using the second constraint, we find that $S_{ii}^{(r;k)} = S_i - \sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)} = S_i - S_{-i}^{(r;k)}$, which can be substituted into retailer i's profit function $\pi_i(\mathbf{S}_i^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$. As a result, retailer i's constrained maximization problem in (17) can be reduced to

$$\max_{S_{ij}^{(r;k)}, j \in C_{-i}(r;k)} \pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)}), \quad \text{s.t.} \sum_{j \in C(r;k)} \Delta\pi_j \geq 0, \tag{18}$$

where retailer i has $k - 1$ decision variables (i.e., $S_{ij}^{(r;k)}$, for $j \in C_{-i}(r;k)$).

4.2.1 Nash Equilibrium

We note from (18) that retailer i's profit function $\pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$ is too complicated and one thus cannot solve the maximization problem to find closed-form solutions. But, we can obtain the property of $\pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$, as shown in the following theorem.

Theorem 3 Retailer i's profit function $\pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$ in (18) is a unimodal function of the retailer's decision variables $S_{ij}^{(r;k)}$, for $j \in C_{-i}(r;k)$. ■

Using the above theorem, we find that the Nash equilibrium-based space allocation decisions for each retailer in the coalition $C(r;k)$ possesses the following properties.

Theorem 4 For the space allocation game for the coalition $C(r;k)$, Nash equilibrium must be unique. Moreover, in Nash equilibrium, retailer $i \in C(r;k)$ allocates a positive space to each of the other retailers, i.e., $S_{ij}^{(r;k)N} > 0$, for $j \in C_{-i}(r;k)$. Such a result means that all retailers $j \in C_{-i}(r;k)$ must take a part of the excess space at retailer i's own site. ■

The above theorem indicates that if in any coalition all retailers should share their spaces each other, which differs from our result for the two-retailer space allocation game in Theorem 1. Specifically, when only two retailers consider the space exchange strategy, one retailer may not decide to allocate any space to the other. That is, the space-exchange strategy may not be successful when only two retailers are involved. However, as Theorem 4 indicates, if three or more retailers cooperate with such a strategy, then each retailer will decide to allocate a part of the space at his own site to each of the other retailers.

4.2.2 Cooperative Game in Characteristic-Function Form

Similar to Section 3.2.2, all retailers in a coalition need a fair allocation that is used to divide these retailers' total profit surplus among them. However, different from Section 3.2.2, we now consider the allocation problem for $n \geq 3$ retailers; thus, we cannot use the generalized Nash bargaining scheme but need alternative game concepts for our analysis in this section. Von Neumann and Morgenstern [39] developed a theory of multi-person games where various subgroups of players might join together to form coalitions. For our n-retailer game where n retailers possibly form different coalition structures to exchange their excess spaces, we construct the space-exchange cooperative game in characteristic-function form by computing the characteristic values of all possible coalitions. For a coalition, the corresponding characteristic value represents the profit surplus that is jointly achieved by all retailers in this coalition, and it is thus calculated as the sum of the profit surpluses of all of these retailers. More specifically,

1. In the empty coalition $\varnothing$, there is no retailer and the profit surplus is certainly zero. Thus, the characteristic value of the empty coalition is $v(\varnothing) = 0$.
2. In a single-retailer coalition $C(i;1) \equiv (i)$, for $i = 1, 2, \ldots, n$, the retailer does not cooperate with any other retailers for the space exchange, and thus, $v(i) = 0$.
3. In the rth k-retailer coalition $C(r;k)$ with $r = 1, 2 \ldots, \binom{n}{k}$ and $2 \leq k \leq n$, the k retailers exchange their excess spaces. Total profit surplus generated by the space exchange is the sum of all k retailers' profit surpluses. That is, $v(C(r;k)) \equiv \sum_{i \in C(r;k)} [\pi_i(\mathbf{S}_i^{(r;k)}; \mathbf{S}_{+i}^{(r;k)}) - \pi_i(S_i)]$, which, using (14) and (16), can be calculated as,

$$v(C(r;k)) = \sum_{i \in C(r;k)} m_i \left[D_{ii}(S_i - S_{-i}^{(r;k)}) - D_{ii}(S_i) \right] + \sum_{i \in C(r;k)} \sum_{j \in C_{-i}(r;k)} m_i D_{ij}(S_{ji}^{(r;k)}).$$

It is interesting—and important—to determine whether or not the above cooperative game in characteristic-function form is *essential* and *superadditive*. An essential cooperative game with n players has the property that $\sum_{i \in N} v(i) < v(N)$ where $N \equiv \{1, 2, \ldots, n\}$; and a game is superadditive if $v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$ for any two *disjoint* coalitions C_1 and C_2 in the n-player game; for details, see, for example, Straffin [36].

Theorem 5 When all retailers exchange their locally-optimal excess spaces that maximize their own profit, we find that the cooperative game in characteristic-function form is essential and superadditive. ■

The above theorem shows that a coalition with more retailers will enjoy a higher profit surplus from the space-exchange strategy. This implies that a greater number of retailers should cooperate, in order to profit more from space exchange.

4.2.3 Allocation of the System-Wide Profit Surplus

As Theorem 5 indicates, the retailers in our n-player ($n \geq 3$) cooperative game in characteristic-function form should have incentives to join the grand coalition $C(n)$. Since there is only one grand coalition and thus $r = 1$ in the coalition $C(r, n)$, for notational simplicity we omit r in this notation for the grand coalition. This happens because the total profit surplus generated by all retailers in $C(n)$—that is the characteristic value of $C(n)$, i.e., $v(C(n))$—is no less than sum of profit surpluses achieved by retailers in all disjoint, less-than-n-retailer, but nonempty coalitions. This means that the grand coalition $C(n)$ is stable if $v(C(n))$ is allocated to all retailers in a fair way. More specifically, if the allocation to each retailer is no smaller than what this retailer would obtain after leaving the grand coalition, then all retailers would be willing to stay in the grand coalition which is thus *stable*. Otherwise, the retailer would leave the grand coalition $C(n)$, which is thus unstable and would disperse.

The Core We now consider the fair allocation of the characteristic value $v(C(n))$ to ensure the stability of the grand coalition $C(n)$. Letting y_i denote the profit surplus allocated to retailer i, for $i \in C(n) = \{1, 2, \ldots, n\}$, we represent a proper allocation scheme by using an n-tuple of numbers $y \equiv (y_1, y_2, \ldots, y_n)$ with the following two properties: (i) *individual rationality*, i.e., $y_i \geq v(i)$, for all $i \in C(n)$; (ii) *collective rationality*, i.e., $\sum_{i \in C(n)} y_i = v(C(n))$. Note that the n-tuple $(y_1, y_2, \ldots, y_n)$ satisfying the above two properties is called an *imputation* for the game $\mathcal{G} = (C(n), v)$; see Straffin [36]. In cooperative game theory there are a number of concepts that could be used for our analysis for the n-player cooperative game in characteristic form. One of the most important concepts is the *core* [18], which is defined as the set of all undominated imputations $(y_1, y_2, \ldots, y_n)$ such that for all coalitions $T \subseteq C(n) = \{1, 2, \ldots, n\}$, we have $\sum_{i \in T} y_i \geq v(T)$.

Theorem 6 When three or more retailers exchange their Nash equilibrium-based excess spaces, the core of our n-retailer cooperative game in characteristic function is non-empty. That is, for this case, the grand coalition $C(n)$ is stable if all retailers implement an allocation scheme in the core. ♦

As the above theorem indicates, any imputation in the non-empty core represents a fair allocation scheme that ensures the stability of the grand coalition $C(n)$. However, since the core is a set of many imputations, one may need the answer to the following question: Which imputation in the core should be applied to the allocation of the system-wide profit surplus $v(C(n))$? Thus, it would be interesting to find a unique allocation solution for our cooperative game, even though some researchers [e.g., Hamlen et al. [19]] believe that a unique solution does not permit any flexibility in management.

The Weighted Shapley Value In the theory of cooperative games, Shapley value [35] represents a unique imputation $(y_1, y_2, \ldots, y_n)$ where the payoffs y_i $(i = 1, 2, \ldots, , n)$ are distributed "fairly" by an outside arbitrator, using the following three axioms—(i) symmetry; (ii) zero allocation to dummy player; and (iii) additivity. However, for our space-exchange game, we cannot use Shapley value to uniquely allocate the maximum profit surplus $v(C(n))$ among n retailers in the grand coalition $C(n)$, because we allow *unequal* bargaining powers of all retailers and thus Axiom (i) may not be satisfied.

In the theory of cooperative games, the concept of "*weighted* Shapley value" can be used to fairly allocate the profit surplus or cost saving among $n \geq 3$ players with different bargaining powers. This concept was introduced by Shapley [34]; and then discussed by a number of economists and mathematicians. But, very few publications used the concept to analyze game problems in the business area. A representative publication is Loehman and Whinston [29], who applied the weighted Shapley value to an allocation problem in accounting. But, in [29], the weighted Shapley value may be outside the core and cannot assure the stability of the grand coalition. Different from [29], we will adopt a new approach to compute the weighted Shapley value that must be in the core. Next, we briefly describe the concept and introduce our approach.

The weighted Shapley value associates a positive *weight* to each of k players who join a k-player coalition and share the total profit surplus (in this k-retailer coalition) according to their weights. For our space-exchange problem, we shall determine the weighted Shapley value by using Harsanyi's procedure [22], where the weighted Shapley value can be calculated in terms of the following weight set,

$$\mathbf{w} \equiv \{\mathbf{w}(r;k), \text{ for } r = 1, 2 \ldots, \binom{n}{k} \text{ and } 2 \leq k \leq n\}, \tag{19}$$

with

$$\mathbf{w}(r;k) \equiv \left\{ w_i(r;k) : 0 \leq w_i(r;k) \leq 1 \text{ and } \sum\nolimits_{i \in C(r;k)} w_i(r;k) = 1, \text{ for } i \in C(r;k) \right\}, \tag{20}$$

where $w_i(r;k)$ denotes retailer i's weight in the coalition $C(r;k)$, in which k retailers share the profit surplus $v(C(r;k))$ according to their weights $w_i(r;k)$ $(i = 1, 2, \ldots, k)$. The weighted Shapley value (allocation) y_i of retailer i (for $i = 1, 2, \ldots, n)$ is equal to the total accumulated residuals allocated to this retailer, i.e.,

$$y_i = \sum\nolimits_{k=1}^{n} \varepsilon_i(k), \quad \text{for } i = 1, 2, \ldots, n. \tag{21}$$

where $\varepsilon_i(k)$ denotes all residuals that are allocated to retailer $i = 1, 2, \ldots, n$ in all possible k-retailer coalitions that retailer i could join, and it is computed as

$$\varepsilon_i(k) = \sum\nolimits_{r=1}^{\binom{n}{k}} \varepsilon_{I_i(r;k)}(r;k) \times \mathbf{1}_{I_i(r;k)>0},$$

where $\varepsilon_{I_i(r;k)}(r;k)$ denotes the residual that is allocated to retailer i who is the $I_i(r;k)$-th player in the coalition $C(r;k)$; $\mathbf{1}_{I_i(r;k)>0} = 1$ when $I_i(r;k) > 0$, and $\mathbf{1}_{I_i(r;k)>0} = 0$ when $I_i(r;k) = 0$.

Hamlen *et al.* [19] showed that there must exist a convex set of some weight sets each yielding a unique weighted Shapley value that is in the core (if the core is non-empty). We denote by Γ the convex set of all weight sets that make the weighted Shapley value to satisfy the core conditions. Because Theorem 6 indicates that the core of our space-exchange game is non-empty, the set Γ must be non-empty, and we can choose any weight set in Γ to find a weighted Shapley value that satisfies the core condition and ensures the stability of the grand coalition $C(n)$.

Next, we focus on the question of which weight set in Γ should be chosen to reflect the retailers' bargaining powers $(\gamma_1, \gamma_2, \ldots, \gamma_n)$. Loehman and Whinston [29] set the weights of all retailers in a coalition to their *relative* bargaining powers in this coalition. Consider the coalition $C(r;k)$, for $r = 1, 2, \ldots, \binom{n}{k}$ and $k = 1, 2, \ldots, n$. If these k retailers have their bargaining powers $(\gamma_1, \gamma_2, \ldots, \gamma_k)$, then the weight $w_i(r;k)$ of retailer i is equal to his *relative* bargaining power $\hat{\gamma}_i(r;k)$, i.e.,

$$w_i(r;k) = \hat{\gamma}_i(r;k) \equiv \gamma_i \Big/ \sum\nolimits_{j=1}^{k} \gamma_j \text{ , for } i = 1, 2, \ldots, k. \tag{22}$$

However, Loehman and Whinston [29] cannot guarantee that the weight set $LW \in \Gamma$ (where LW is the set of Loehman and Whinston's relative bargaining powers). Thus, we suggest an approach to find the weights that reflect bargaining powers $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ and also satisfy the core conditions.

Because each weight set $\mathbf{w}$ in Γ must yield a weighted Shapley value that is in the core, there must exist a weight set $\hat{\mathbf{w}}$ (in Γ) that is the *closest* to LW [29]. More specifically, if LW is in Γ, then we select it as the weight set $\hat{\mathbf{w}}$, i.e., $\hat{\mathbf{w}} = LW$, as illustrated in Figure 1(a); otherwise, if LW is not in Γ, then we use the weight set (in Γ) that is the closest to LW, as illustrated in Figure 1(b). It is easy to justify that the set $\hat{\mathbf{w}}$ reflects the bargaining powers, because, as the bargaining powers $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ change, then LW also changes and the set $\hat{\mathbf{w}}$ accordingly changes.

In order to find $\hat{\mathbf{w}}$, we need to minimize the distance between $\mathbf{w}$ and LW, subject to the weighted Shapley value in terms of the set $\mathbf{w}$ satisfies the core conditions. The distance between

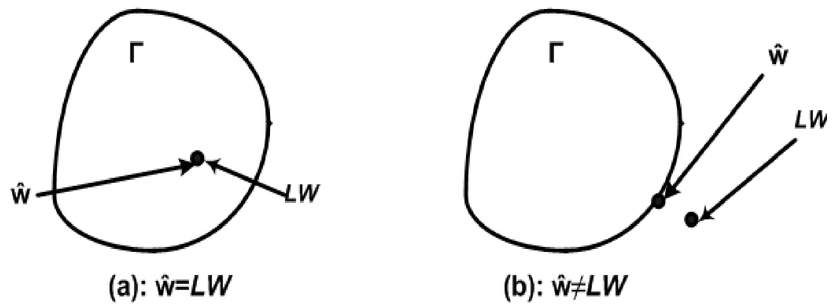

Figure 1: The calculation of the weight set $\hat{\mathbf{w}}$ that is in Γ and thus makes the corresponding weighted Shapley value to satisfy the core conditions and also reflects the retailers' bargaining powers.

$\mathbf{w}$ and LW is denoted by $\|\mathbf{w} - LW\|$, which is calculated as

$$\|\mathbf{w} - LW\| \equiv \sum_{k=1}^{n} \sum_{r=1}^{\binom{n}{k}} \sum_{i \in C(r;k)} [w_i(r;k) - \hat{\gamma}_i(r;k)]^2. \tag{23}$$

Hence, we can use $\hat{\mathbf{w}}$ to find a unique weighted Shapley value that satisfies the core conditions and reflects the retailers' bargaining powers.

Theorem 7 The unique weighted Shapley value (allocation) that ensures the stability of the grand coalition is $\hat{y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$, where $\hat{y}_i$ $(i = 1, 2, \ldots, n)$ is computed by using (21) with the weight set $\hat{\mathbf{w}}$. The weight set $\hat{\mathbf{w}}$ is the unique solution of the following constrained non-linear programming problem:

$$\begin{array}{ll} \min_{\mathbf{w}} & \|\mathbf{w} - LW\| \\ \text{s.t.} & \sum_{i \in C(r;k)} y_i \geq v(C(r;k)), \text{ for } r = 1, 2, \ldots, \binom{n}{k} \text{ and } k = 1, 2, \ldots, n; \\ & \sum_{i \in C(r;k)} w_i(r;k) = 1, \text{ for } r = 1, 2, \ldots, \binom{n}{k} \text{ and } k = 1, 2, \ldots, n. \end{array} \tag{24}$$

[Note that in this problem we use (21) to obtain y_i $(i = 1, 2, \ldots, n)$, which is a function of a given weight set $\mathbf{w}$.] ■

The weighted Shapley value computed by using Theorem 7 must be in the core, because of the following reason: The weight set $\hat{\mathbf{w}}$ is obtained by solving the constrained minimization problem in (24). Thus, the weighted Shapley value $\hat{y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$ in terms of $\hat{\mathbf{w}}$ must satisfy the constraint $\sum_{i \in C(r;k)} \hat{y}_i \geq v(C(r;k))$, for $r = 1, 2, \ldots, \binom{n}{k}$ and $k = 1, 2, \ldots, n$. That is, for all coalitions $T \subseteq C(n) = \{1, 2, \ldots, n\}$, we find that $\sum_{i \in T} \hat{y}_i \geq v(T)$, which means that the weighted Shapley value is in the core.

In order to justify that the weighted Shapley value reflects the bargaining powers, we consider the following case: Assuming that retailer i has more bargaining power (i.e., the value of γ_i increases) but other retailers' bargaining powers are unchanged, we find from (22) that, in any coalition $C(r;k)$ that retailer i joins, this retailer's relative bargaining power increases, that is, $\hat{\gamma}_i(r;k)$ increases. As Theorem 7 shows, we find that retailer i's allocation weight

$\hat{w}_i(r;k)$ in the set $\hat{\mathbf{w}}$ must also increase because we find $\hat{\mathbf{w}}$ by minimizing $\|\mathbf{w} - LW\|$ which is defined by (23). Moreover, we note that, when LW is not in the core, the weight set $\hat{\mathbf{w}}$ that are found by using our approach must be on the edge of the convex set Γ.

4.2.4 Two Numerical Examples for the Space Exchange Analysis with Three Retailers

In order to illustrate our analysis, we now consider a three-retailer space exchange problem and present two numerical examples—one with LW in the set Γ and one with LW outside the set Γ.

Example 1 We consider three retailers (i.e., retailers 1, 2, and 3). The three retailers' parameter values are supposed as in Table 2, where some parameters (e.g., m_i, h_i, and γ_i, for $i = 1, 2$) for retailers 1 and 2 are assumed to take the numerical values in Section 3.2.3. As discussed previously, we need to consider each possible coalition and calculate corresponding space allocation decisions in Nash equilibrium. Since a single retailer cannot implement the space exchange strategy, we should solve four space exchange games for three two-player games and one three-player game.

m_1	m_2	m_3	h_1	h_2	h_3	S_1	S_2	S_3	γ_1	γ_2	γ_3	δ_{12}^1	δ_{12}^2	δ_{12}^3	δ_{13}^1	δ_{13}^2
15	25	9	4	6	5	100	60	50	3	8	6	0.1	0.2	0.1	0.1	0.1
δ_{13}^3	δ_{23}^1	δ_{23}^2	δ_{23}^3	α_{11}^2	$\hat{\alpha}_{11}^2$	α_{11}^3	$\hat{\alpha}_{11}^3$	α_{12}^2	$\hat{\alpha}_{12}^2$	α_{12}^3	$\hat{\alpha}_{12}^3$	α_{13}^2	$\hat{\alpha}_{13}^2$	α_{13}^3	$\hat{\alpha}_{13}^3$	α_{21}^1
0.2	0.2	0.2	0.1	4	0.3	1	0.4	1	0.2	3	0.3	2	0.1	1	0.1	3
$\hat{\alpha}_{21}^1$	α_{21}^3	$\hat{\alpha}_{21}^3$	α_{22}^1	$\hat{\alpha}_{22}^1$	α_{22}^3	$\hat{\alpha}_{22}^3$	α_{23}^1	$\hat{\alpha}_{23}^1$	α_{23}^3	$\hat{\alpha}_{23}^3$	α_{31}^1	$\hat{\alpha}_{31}^1$	α_{31}^2	$\hat{\alpha}_{31}^2$	α_{32}^1	$\hat{\alpha}_{32}^1$
0.3	4	0.2	6	0.4	3	0.3	5	0.2	4	0.1	4	0.1	3	0.1	3	0.2
α_{32}^2	$\hat{\alpha}_{32}^2$	α_{33}^1	$\hat{\alpha}_{33}^1$	α_{33}^2	$\hat{\alpha}_{33}^2$	β_1^1	β_1^2	β_1^3	β_2^1	β_2^2	β_2^3	β_3^1	β_3^2	β_3^3		
4	0.3	2	0.1	3	0.2	0.5	0.4	0.3	0.3	0.4	0.6	0.4	0.4	0.5		

Table 2: The parameter values in the numerical example for the three-retailer decentralized case with LW in the set Γ.

For the coalition $\{1, 2\}$ where only retailers 1 and 2 exchange their spaces, we find that $S_{11}^{\{1,2\}N} = 82.68$ sq. ft., $S_{12}^{\{1,2\}N} = 17.32$ sq. ft.; $S_{21}^{\{1,2\}N} = 11.93$ sq. ft., and $S_{22}^{\{1,2\}N} = 48.07$ sq. ft., as given in Section 3.2.3. For the coalition $\{1, 3\}$ where only retailers 1 and 3 exchange their spaces, we find that $S_{11}^{\{1,3\}N} = 38.89$ sq. ft., $S_{13}^{\{1,3\}N} = 61.11$ sq. ft.; $S_{31}^{\{1,3\}N} = 6.56$ sq. ft., and $S_{33}^{\{1,3\}N} = 43.45$ sq. ft. For the coalition $\{2, 3\}$ where only retailers 2 and 3 exchange their spaces, we find that $S_{22}^{\{2,3\}N} = 39.41$ sq. ft., $S_{23}^{\{2,3\}N} = 20.59$ sq. ft.; $S_{33}^{\{2,3\}N} = 42.77$ sq. ft., and $S_{32}^{\{2,3\}N} = 7.23$ sq. ft. For the grand coalition $\{1, 2, 3\}$ where three retailers exchange their spaces each other, we compute the Nash equilibrium-based space allocation decisions (in sq. ft.) as follows:

$S_{11}^{\{1,2,3\}N}$	$S_{12}^{\{1,2,3\}N}$	$S_{13}^{\{1,2,3\}N}$	$S_{21}^{\{1,2,3\}N}$	$S_{22}^{\{1,2,3\}N}$	$S_{23}^{\{1,2,3\}N}$	$S_{31}^{\{1,2,3\}N}$	$S_{32}^{\{1,2,3\}N}$	$S_{33}^{\{1,2,3\}N}$
77.56	4.17	18.27	2.83	52.74	4.43	11.38	5.55	33.07

Next, we use our formulas in Section 4.2.2 to construct a cooperative game in characteristic-function form as,

$$v(\varnothing) = v(1) = v(2) = v(3) = 0,$$
$$v(12) = 1108.11, \quad v(13) = 646.46, \quad v(23) = 3271.81, \quad v(123) = 3565.28.$$

Since $v(ij) > v(i) + v(j)$ and $v(123) > v(ij) + v(k)$ for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, the game is essential and superadditive with a nonempty core. In order to find a unique weighted Shapley value, we first examine whether or not Loehman and Whinston's weight set LW [29] is in the set Γ. To do so, we need to follow Harsanyi's procedure to find the weighted Shapley value in terms of LW as follows: $y_1 = 259.86$, $y_2 = 1987.93$, $y_3 = 1317.49$. It is easy to verify that $y_i > v(i) = 0$ and $y_i + y_j > v(ij)$, for $i, j = 1, 2, 3$ and $i \neq j$; and $y_1 + y_2 + y_3 = v(123)$. This means that the weighted Shapley value in terms of the weight set LW is in the core; so, it is used for our allocation. Note that we can also solve the nonlinear problem in Theorem 7 to find $\hat{\mathbf{w}} = LW$. ■

In the above example, we find that, when we use the weight set LW suggested by Loehman and Whinston [29], the weighted Shapley value is in the core. Next, we provide another example in which LW is not in the core and we thus have to use our method in Theorem 7 to find a unique weighted Shapley value.

Example 2 We still consider the three retailers of Example 1 but change some parameter values to the following: $S_2 = 70$ sq. ft., $\gamma_2 = 5$, and $\gamma_3 = 4$. Similar to Example Example 1, we can compute the space allocation decisions in Nash equilibrium for all possible coalitions, and construct a cooperative game in the characteristic-function form as

$$v(\varnothing) = v(1) = v(2) = v(3) = 0,$$
$$v(12) = 1129.96, \quad v(13) = 646.46, \quad v(23) = 3317.48, \quad v(123) = 3639.38,$$

which is essential and superadditive with a nonempty core, because $v(ij) > v(i) + v(j)$ and $v(123) > v(ij) + v(k)$ for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. But, using Loehman and Whinston's weight set LW, we compute the weighted Shapley value as $y_1 = 337.15$, $y_2 = 1943.22$, and $y_3 = 1359.01$. We find that $y_2 + y_3 = 3302.23 < v(23) = 3317.48$; this means that the weighted Shapley value in terms of the weight set LW is *not* in the core. Thus, we use our method in Theorem 7 to find a unique weighted Shapley value as $\hat{y}_1 = 322.26$, $\hat{y}_2 = 1954.54$, and $\hat{y}_3 = 1362.58$, which satisfies the core conditions and also reflects the bargaining powers of three retailers. For details regarding our computation, see online Appendix B. ■

5 Summary and Concluding Remarks

This paper is motivated by the practice of Waitrose and Boots/Welcome Break (and also, Tim Hortons and Cold Stone Creamery) where these retailers exchange their excess spaces to increase their profits. We consider a two-retailer problem and an n-retailer problem with $n \geq 3$. For each problem, we calculate the space allocation decisions in Nash equilibrium, and use cooperative game theory to allocate the system-wide profit surplus resulting from space exchange between two or among three or more retailers to assure that the retailers with different bargaining powers are willing to cooperate for the space exchange. Specifically, we use the generalized Nash bargaining scheme to allocate the system-wide profit surplus for the two-retailer case, and apply the weighted Shapley value to divide the surplus for the $n-$retailer case ($n \geq 3$). We proposed a new approach to compute a unique weighted Shapley value that assures the stability of the grand coalition and reflects the retailers' relative bargaining powers.

In addition to our contributions to the literature, as summarized in Section 2.3, there are six major managerial insights as follows:

1. For the two-retailer case, we find that, in Nash equilibrium, the retailer with a higher value of the ratio of the operating cost to the profit margin (i.e., h_i/m_i, for $i = 1, 2$) should secure less host space and allocate more to the other retailer.
2. For the two-retailer case, each retailer may decide to secure all space at his own site. That is, if only two retailers are involved, then they may or may not decide to exchange their spaces.
3. Under the allocation scheme for the two-retailer case, a retailer should make a payment transfer to the other retailer. The side payment depends on the comparison between two retailers' relative profit surpluses and their relative bargaining powers.
4. For the n-retailer case with $n \geq 3$, we show that, in Nash equilibrium, each retailer should allocate a positive part of his space to each of the other retailers. That is, different from the two-retailer case, in the n-retailer case any two retailers will exchange their spaces.
5. When three or more retailers cooperate for the space exchange, a positive system-wide profit surplus will be generated. Moreover, a fair allocation scheme must exist such that all retailers can profit more from the space exchange and thus willing to exchange their spaces.
6. The space exchange by more retailers will result in a higher system-wide profit surplus and thus a higher allocation to each retailer under a fair scheme. This means that a greater number of retailers should cooperate, in order to profit more from the space exchange.

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Online Supplements
"Cooperative Game Analysis of Retail Space-Exchange Problems"
M. Leng, M. Parlar, D. Zhang

Appendix A Proofs

Proof of Theorem 1. Temporarily ignoring constraints in (7), we first find the optimal solution that maximizes $\pi_i(S_{ii}; S_{jj})$ in (5), where $i, j = 1, 2$ and $i \neq j$. Taking the first- and second-order derivatives of $\pi_i(S_{ii}; S_{jj})$ w.r.t. S_{ii} yields

$$\begin{aligned}\frac{d\pi_i(S_{ii}; S_{jj})}{dS_{ii}} &= m_i\beta_i^i S_{ii}^{\beta_i^i-1}[\alpha_{ii} + \hat{\alpha}_{ii}(S_i - S_{ii})^{\delta_{i\ell}^i}] - m_i\hat{\alpha}_{ii}\delta_{i\ell}^i S_{ii}^{\beta_i^i}(S_i - S_{ii})^{\delta_{i\ell}^i-1} - h_i, \qquad (25)\\ \frac{d^2\pi_i(S_{ii}; S_{jj})}{dS_{ii}^2} &= m_i\beta_i^i(\beta_i^i - 1)S_{ii}^{\beta_i^i-2}[\alpha_{ii} + \hat{\alpha}_{ii}(S_i - S_{ii})^{\delta_{i\ell}^i}] - 2m_i\beta_i^i\hat{\alpha}_{ii}\delta_{i\ell}^i S_{ii}^{\beta_i^i-1}(S_i - S_{ii})^{\delta_{i\ell}^i-1}\\ &\quad + m_i\hat{\alpha}_{ii}\delta_{i\ell}^i(\delta_{i\ell}^i - 1)S_{ii}^{\beta_i^i}(S_i - S_{ii})^{\delta_{i\ell}^i-2},\end{aligned}$$

which is negative because $0 < \beta_i^i, \delta_{i\ell}^i < 1$. Therefore, retailer i's profit function $\pi_i(S_{ii}; S_{jj})$ is strictly concave in S_{ii}. This means that we can uniquely compute retailer i's optimal decision $\hat{S}_{ii}$ that maximizes $\pi_i(S_{ii}; S_{jj})$, by equating $d\pi_i(S_{ii}; S_{jj})/dS_{ii}$ in (25) to zero and solving the resulting equation for S_{ii}. However, due to the complexity of $d\pi_i(S_{ii}; S_{jj})/dS_{ii}$, we cannot find the analytic solution for $\hat{S}_{ii}$ but can conclude that $\hat{S}_{ii}$ uniquely satisfies

$$\alpha_{ii}\beta_i^i S_{ii}^{\beta_i^i-1} + \hat{\alpha}_{ii}S_{ii}^{\beta_i^i-1}(S_i - S_{ii})^{\delta_{i\ell}^i-1}[\beta_i^i(S_i - S_{ii}) - \hat{\alpha}_{ii}\delta_{i\ell}^i S_{ii}] = h_i/m_i.$$

The above is independent of S_{jj}, which is attributed to the fact that, under the space-exchange strategy, each retailer can only determine the allocation of his own space. It thus follows that Nash equilibrium (S_{11}^N, S_{22}^N) for the simultaneous-move game must be unique. Next, we consider the constraints in (5) to calculate S_{11}^N and S_{22}^N.

1. We first consider the constraint $\Delta\pi_1 + \Delta\pi_2 \geq 0$, where, using (6),

$$\begin{aligned}\Delta\pi_1 + \Delta\pi_2 &= m_1[D_{11}(S_{11}) - D_{11}(S_1)] + m_2D_{21}(S_1 - S_{11})\\ &\quad + m_2[D_{22}(S_{22}) - D_{22}(S_2)] + m_1D_{12}(S_2 - S_{22}),\end{aligned}$$

which is, of course, zero when $S_{ii} = S_i$ $(i = 1, 2)$. However, we cannot draw any result for the sign of $\Delta\pi_1 + \Delta\pi_2$, when $S_{ii} < S_i$. Next, assuming that $S_{ii} < S_i$, we examine the sign of the following term (in $\Delta\pi_1 + \Delta\pi_2$)

$$m_1[D_{11}(S_{11}) - D_{11}(S_1)] + m_2D_{21}(S_1 - S_{11}), \qquad (26)$$

which depends on whether or not $D_{11}(S_{11})$ is greater than $D_{11}(S_1)$. We learn from (3) that $D_{11}(S_{11}) = S_{11}^{\beta_1^1}[\alpha_{11} + \hat{\alpha}_{11}(S_1 - S_{11})^{\delta_{12}^1}]$. To compare $D_{11}(S_{11})$ and $D_{11}(S_1)$, we calculate the first- and second-order derivatives of the function $D_{11}(S_{11})$ w.r.t. S_{11} as follows:

$$\begin{aligned}
\frac{d[D_{11}(S_{11})]}{dS_{11}} &= \beta_1^1 S_{11}^{\beta_1^1-1}[\alpha_{11}+\hat{\alpha}_{11}(S_1-S_{11})^{\delta_{12}^1}] - \hat{\alpha}_{11}\delta_{12}^1 S_{11}^{\beta_1^1}(S_1-S_{11})^{\delta_{12}^1-1}, \\
\frac{d^2[D_{11}(S_{11})]}{dS_{11}^2} &= \beta_1^1(\beta_1^1-1)S_{11}^{\beta_1^1-2}[\alpha_{11}+\hat{\alpha}_{11}(S_1-S_{11})^{\delta_{12}^1}] \\
&\quad -2\hat{\alpha}_{11}\beta_1^1\delta_{12}^1 S_{11}^{\beta_1^1-1}(S_1-S_{11})^{\delta_{12}^1-1} + \hat{\alpha}_{11}\delta_{12}^1(\delta_{12}^1-1)S_{11}^{\beta_1^1}(S_1-S_{11})^{\delta_{12}^1-2},
\end{aligned}$$

which is negative because $0 < \beta_1^1, \delta_{12}^1 < 1$. This means that the function $D_{11}(S_{11})$ is a strictly concave function of S_{11}. Moreover, we can easily find that

$$D_{11}(0)=0,\ \left.\frac{d[D_{11}(S_{11})]}{dS_{11}}\right|_{S_{11}=0} = +\infty;\ D_{11}(S_1) = \alpha_{11}S_{11}^{\beta_1^1},\ \left.\frac{d[D_{11}(S_{11})]}{dS_{11}}\right|_{S_{11}=S_1} = -\infty.$$

We plot Figure 2 to visionally show the function $D_{11}(S_{11})$, and find that there must exist a point $z_1 \in (0, S_1)$ such that, $\forall S_{11} \in [z_1, S_1)$ or $\forall S_{12} \in [0, S_1 - z_1)$, $D_{11}(S_{11}) \geq D_{11}(S_1)$.

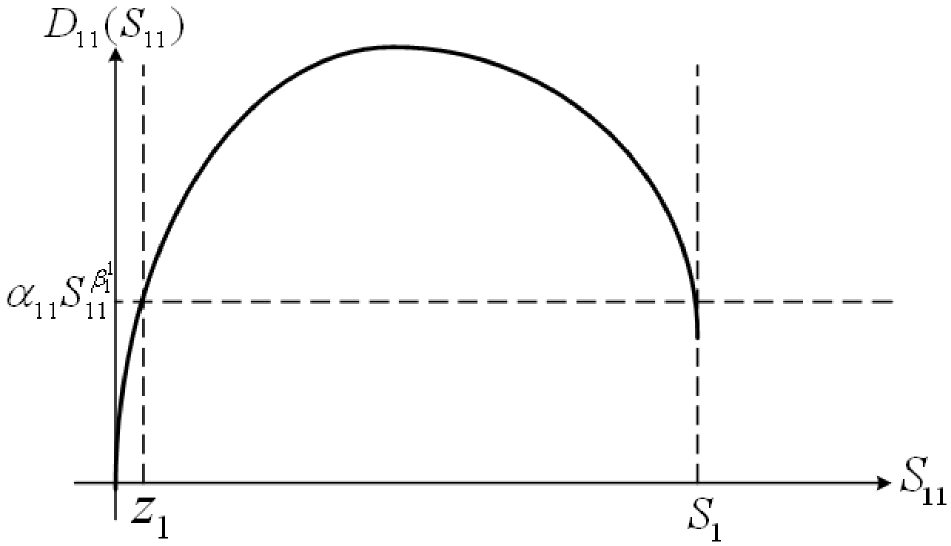

Figure 2: The shape of the function $D_{11}(S_{11})$ where $S_{11} = S_1 - S_{12}$.

From the above, we find that, if $S_{11} \geq z_1$, then $D_{11}(S_{11}) \geq D_{11}(S_1)$ and the term (26) in $\Delta\pi_1 + \Delta\pi_2$ must be non-negative. Similarly, there also must exist the point z_2. If $S_{22} \geq z_2$, then $D_{22}(S_{22}) \geq D_{22}(S_2)$. Thus, if $\hat{S}_{ii}$ $(i = 1, 2)$—which maximizes $\pi_i(S_{ii}; S_{jj})$, where $j = 1, 2$ and $j \neq i$—is greater than or equal to z_i, then retailer i should choose $\hat{S}_{ii}$ as his optimal solution when the other constraint (i.e., $0 \leq S_{ii} \leq S_i$) is ignored. Otherwise, retailer i's optimal solution should z_i.

2. We then consider the constraint $0 \leq S_{ii} \leq S_i$, under which if $\hat{S}_{ii} \leq S_i$, then retailer i should choose $\hat{S}_{ii}$ as his optimal solution when the first constraint (i.e., $\Delta\pi_1 + \Delta\pi_2 \geq 0$) is ignored. Otherwise, retailer i's optimal solution should S_i.

Noting that $z_i \in (0, S_i)$ (for $i = 1, 2$), we can use the above discussion to find two retailers' space allocation decisions in Nash equilibrium as $\min(\max(\hat{S}_{ii}, z_i), S_i)$, which can be written as given in this theorem. ■

Proof of Theorem 2. Let $G(y_1, y_2)$ denote the objective function in (11), i.e., $G(y_1, y_2) \equiv y_1^{\gamma_1} y_2^{\gamma_2}$. Since $y_1 + y_2 = \Delta\pi_1 + \Delta\pi_2$, we substitute $y_2 = (\Delta\pi_1 + \Delta\pi_2) - y_1$ into $G(y_1, y_2)$, and obtain the objective function only in terms of the variable y_1 as $G(y_1) = y_1^{\gamma_1}[(\Delta\pi_1 + \Delta\pi_2) - y_1]^{\gamma_2}$. Next, we temporarily ignore the non-negative constraints $y_1 \geq 0$ and $y_2 \geq 0$, and compute the optimal solution y_1^* that maximizes $G(y_1)$.

Because two retailers obtain positive allocations when $\Delta\pi_1 + \Delta\pi_2 > 0$, we find that $G(y_1) > 0$, and we can thus compute y_1^* by simply maximizing the logarithm of $G(y_1)$ which is written as

$$\ln[G(y_1)] = \gamma_1 \ln y_1 + \gamma_2 \ln[(\Delta\pi_1 + \Delta\pi_2) - y_1].$$

Taking the first- and second-order derivatives of $\ln[G(y_1)]$ w.r.t. y_1, we have

$$\begin{aligned}
\frac{d\ln[G(y_1)]}{dy_1} &= \frac{\gamma_1}{y_1} - \frac{\gamma_2}{(\Delta\pi_1 + \Delta\pi_2) - y_1}, \\
\frac{d^2\ln[G(y_1)]}{dy_1^2} &= -\frac{\gamma_1}{y_1^2} - \frac{\gamma_2}{[(\Delta\pi_1 + \Delta\pi_2) - y_1]^2} \leq 0,
\end{aligned}$$

which implies that $\ln[G(y_1)]$ is concave in y_1. We solve $d\ln[G(y_1)]/dy_1 = 0$ and find the optimal solution y_1^* as shown in (12), and then compute $y_2^* = (\Delta\pi_1 + \Delta\pi_2) - y_1^*$ which can be simplified to the result in (12). It is easy to justify that $y_i^* \geq 0$ $(i = 1, 2)$ and the nonnegative constraints in (11) are satisfied. This proves the theorem. ■

Proof of Theorem 3. According to Section 4.1, we write the retailer's profit function as,

$$\begin{aligned}
& \pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)}) \\
= \; & m_i \sum_{j\in C(r;k)} D_{ij}(S_{ji}^{(r;k)}) - \sum_{j\in C(r;k)} h_j S_{ji}^{(r;k)} \\
= \; & m_i \left\{ \left(S_i - \sum_{j\in C_{-i}(r;k)} S_{ij}^{(r;k)} \right)^{\beta_i^i} \prod\nolimits_{\ell\in C_{-i}(r;k)} [\alpha_{ii}^\ell + \hat{\alpha}_{ii}^\ell (S_{i\ell}^{(r;k)})^{\delta_{i\ell}^i}] \right\} \\
& -h_i \left(S_i - \sum_{j\in C_{-i}(r;k)} S_{ij}^{(r;k)} \right) \\
& +m_i \sum_{j\in C_{-i}(r;k)} \left\{ (S_{ji}^{(r;k)})^{\beta_i^j} \prod\nolimits_{\ell\in C_{-i}(r;k)} [\alpha_{ij}^\ell + \hat{\alpha}_{ij}^\ell (S_{j\ell}^{(r;k)})^{\delta_{i\ell}^j}] \right\} - \sum_{j\in C_{-i}(r;k)} h_j S_{ji}^{(r;k)}, \quad (27)
\end{aligned}$$

where retailer i's decision variables appear only in the first two terms. We denote the first two terms in (27) by Z; that is,

$$Z \equiv m_i \hat{Z} - h_i(S_i - \sum_{j\in C_{-i}(r;k)} S_{ij}^{(r;k)}), \quad (28)$$

where

$$\hat{Z} \equiv \hat{Z}_1^{\beta_i^i} \prod\nolimits_{j\in C_{-i}(r;k)} \hat{Z}_j \quad (29)$$

with

$$\hat{Z}_1 \equiv S_i - \sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)} \quad \text{and} \quad \hat{Z}_j \equiv \alpha_{ii}^j + \hat{\alpha}_{ii}^j (S_{ij}^{(r;k)})^{\delta_{ij}^i}, \text{ for } j \in C_{-i}(r;k).$$

We begin by investigating the property of Z. Taking the logarithm of $\hat{Z}$, we find that

$$\xi \equiv \ln(\hat{Z}) = \beta_i^i \ln(\hat{Z}_1) + \sum_{j \in C_{-i}(r;k)} \ln(\hat{Z}_j). \tag{30}$$

Partially differentiating ξ once and twice w.r.t. $S_{ij}^{(r;k)}$ ($j \in C_{-i}(r;k)$) yields,

$$\frac{\partial \xi}{\partial S_{ij}^{(r;k)}} = -\frac{\beta_i^i}{S_i - \sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)}} + \frac{\hat{\alpha}_{ii}^j \delta_{ij}^i (S_{ij}^{(r;k)})^{\delta_{ij}^i - 1}}{\alpha_{ii}^j + \hat{\alpha}_{ii}^j (S_{ij}^{(r;k)})^{\delta_{ij}^i}},$$

and

$$\begin{aligned}\frac{\partial^2 \xi}{\partial (S_{ij}^{(r;k)})^2} \quad &= \quad -\frac{\beta_i^i}{\left[S_i - \sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)}\right]^2} - \frac{\hat{\alpha}_{ii}^j \delta_{ij}^i (1 - \delta_{ij}^i)(S_{ij}^{(r;k)})^{\delta_{ij}^i - 2}}{\alpha_{ii}^j + \hat{\alpha}_{ii}^j (S_{ij}^{(r;k)})^{\delta_{ij}^i}} \\ &\quad -\frac{2(\hat{\alpha}_{ii}^j \delta_{ij}^i)^2 (S_{ij}^{(r;k)})^{2(\delta_{ij}^i - 1)}}{\left[\alpha_{ii}^j + \hat{\alpha}_{ii}^j (S_{ij}^{(r;k)})^{\delta_{ij}^i}\right]^2},\end{aligned} \tag{31}$$

which is negative because $0 < \delta_{ij}^i < 1$. To find whether or not the function ξ is jointly concave in $k-1$ variables $S_{ij}^{(r;k)}$, for $j \in C_{-i}(r;k)$, we also need to examine the signs of leading principle minors in the corresponding Hessian matrix. We calculate the second cross derivative of ξ w.r.t. $S_{ij}^{(r;k)}$ and $S_{i\ell}^{(r;k)}$ ($j, \ell \in C_{-i}(r;k)$ and $j \neq \ell$) as follows:

$$\frac{\partial^2 \xi}{\partial S_{ij}^{(r;k)} \partial S_{i\ell}^{(r;k)}} = -\Lambda_0, \quad \text{where } \Lambda_0 \equiv \frac{\beta_i^i}{\left[S_i - \sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)}\right]^2}. \tag{32}$$

Assuming without loss of generality that in the coalition $C(r;k)$, $j \in C_{-i}(r;k) = \{1, 2, \ldots, k-1\}$ and $i = k$, we can construct the Hessian matrix as,

$$H = \begin{bmatrix} \frac{\partial^2 \xi}{\partial (S_{i1}^{(r;k)})^2} & -\Lambda_0 & \cdots & -\Lambda_0 \\ -\Lambda_0 & \frac{\partial^2 \xi}{\partial (S_{i2}^{(r;k)})^2} & \cdots & -\Lambda_0 \\ \cdots & \cdots & \cdots & \cdots \\ -\Lambda_0 & -\Lambda_0 & \cdots & \frac{\partial^2 \xi}{\partial (S_{i,k-1}^{(r;k)})^2} \end{bmatrix}, \tag{33}$$

where $\partial^2 \xi / \partial (S_{ij}^{(r;k)})^2$ and Λ_0 are given as in (31) and (32), respectively. We also find from (31) and (32) that

$$\frac{\partial^2 \xi}{\partial (S_{ij}^{(r;k)})^2} = \frac{\partial^2 \xi}{\partial S_{ij}^{(r;k)} \partial S_{i\ell}^{(r;k)}} - \frac{\hat{\alpha}_{ii}^j \delta_{ij}^i (1 - \delta_{ij}^i)(S_{ij}^{(r;k)})^{\delta_{ij}^i - 2}}{\alpha_{ii}^j + \hat{\alpha}_{ii}^j (S_{ij}^{(r;k)})^{\delta_{ij}^i}} - \frac{2(\hat{\alpha}_{ii}^j \delta_{ij}^i)^2 (S_{ij}^{(r;k)})^{2(\delta_{ij}^i - 1)}}{\left[\alpha_{ii}^j + \hat{\alpha}_{ii}^j (S_{ij}^{(r;k)})^{\delta_{ij}^i}\right]^2}.$$

Hence, the Hessian matrix can be transformed to the following reduced row–echelon form:

$$H' = \begin{bmatrix} -\Lambda_{i1} & 0 & \cdots & \Lambda_{i,k-1} \\ 0 & -\Lambda_{i2} & \cdots & \Lambda_{i,k-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\Lambda_{i,k-1}\left[1 + \Lambda_0 \times \sum_{j \in C_{-i}(r;k)} \left(\frac{1}{\Lambda_{ij}}\right)\right] \end{bmatrix},$$

where

$$\Lambda_{ij} = \frac{\hat{\alpha}_{ii}^{j}\delta_{ij}^{i}(1-\delta_{ij}^{i})(S_{ij}^{(r;k)})^{\delta_{ij}^{i}-2}}{\alpha_{ii}^{j} + \hat{\alpha}_{ii}^{j}(S_{ij}^{(r;k)})^{\delta_{ij}^{i}}} + \frac{2(\hat{\alpha}_{ii}^{j}\delta_{ij}^{i})^2(S_{ij}^{(r;k)})^{2(\delta_{ij}^{i}-1)}}{\left[\alpha_{ii}^{j} + \hat{\alpha}_{ii}^{j}(S_{ij}^{(r;k)})^{\delta_{ij}^{i}}\right]^2} > 0, \quad \text{for } j =\in C_{-i}(r;k).$$

We find that the sign of the $i-$th leading principal minor of the reduced matrix H' is $(-1)^i$, which implies that the Hessian matrix H in (33) is negative definite. Noting that all diagonal elements of H $(\partial^2\xi/\partial(S_{ij}^{(r;k)})^2)$ are negative, we conclude that the function ξ in (30) is jointly concave in $S_{ij}^{(r;k)}$, for $j \in C_{-i}(r;k)$.

It follows from the above that $\hat{Z}$ in (29) is a log-concave function of $S_{ij}^{(r;k)}$ $(j \in C_{-i}(r;k))$. Recall that the maximization of $\pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$ is equivalent to the maximization of Z in (28), where a linear function of $\hat{Z}$ and decision variables $S_{ij}^{(r;k)}$ $(j \in C_{-i}(r;k))$. Therefore, retailer i's profit function is a unimodal function of his decision variables. ■

Proof of Theorem 4. We first show the uniqueness of Nash equilibrium. Note from Theorem 3 that each retailer's profit function is unimodal of his decision variables. For this proof, we should investigate the constraint $\sum_{j \in C(r;k)} \Delta\pi_j \geq 0$, which means that all retailers can jointly benefit from space exchange. That is, if, given $S_{ji}^{(r;k)}$ for $j \in C_{-i}(r;k)$, retailer i's optimal solutions $S_{ij}^{(r;k)*}$ that maximize $\pi_i(\{S_i - S_{-i}^{(r;k)}\} \cup \mathbf{S}_{-i}^{(r;k)}; \mathbf{S}_{+i}^{(r;k)})$ are greater than zero, then all retailers are better off from sharing their spaces and the constraint $\sum_{j \in C(r;k)} \Delta\pi_j \geq 0$ is redundant.

We now show that $S_{ij}^{(r;k)*}$ $(\forall j \in C_{-i}(r;k))$ must be greater than zero. Note that since $\sum_{j \in C_{-i}(r;k)} S_{ij}^{(r;k)*} = S_{-i}^{(r;k)*}$, there must exist $j \in C_{-i}(r;k)$ such that $S_{ij}^{(r;k)*} > 0$. We assume that retailer ζ's optimal solution $S_{i\zeta}^{(r;k)*} = 0$ $(\zeta \in C_{-i}(r;k))$, which means that this retailer does not operate in retailer i's excess space. We compare the current profit $\pi_{-i}^{(r;k)}$ in (16) (when $S_{i\zeta}^{(r;k)*} = 0$) with that—denoted by $\tilde{\pi}_{-i}^{(r;k)} \equiv \sum_{j \in C_{-i}(r;k)} m_j \tilde{D}_{ji}(\tilde{S}_{ij}^{(r;k)}) - h_i S_{-i}^{(r;k)*}$—for the following case: Retailer j allocates a positive space A—which is greater than zero but smaller than $S_{ij}^{(r;k)*}$, i.e., $0 < A < S_{ij}^{(r;k)*}$—to retailer ζ who then operates at the space A. Note that, for retailer $\zeta \in C_{-i}(r;k)$, $\tilde{S}_{i\zeta}^{(r;k)} = A$; for retailer $j \in C_{-i}(r;k)$, $\tilde{S}_{ij}^{(r;k)} = S_{ij}^{(r;k)*} - A$; for other retailers $\ell \in C_{-i}(r;k)\backslash\{j,\zeta\}$, $\tilde{S}_{i\ell}^{(r;k)} = S_{i\ell}^{(r;k)*}$.

We calculate the difference between $\tilde{\pi}_{-i}^{(r;k)}$ and $\pi_{-i}^{(r;k)}$ as,

$$\tilde{\pi}_{-i}^{(r;k)} - \pi_{-i}^{(r;k)} = \sum_{j \in C_{-i}(r;k)} m_j[\tilde{D}_{ji}(\tilde{S}_{ij}^{(r;k)}) - D_{ji}(S_{ij}^{(r;k)})]. \tag{34}$$

It easily follows that $\tilde{D}_{\zeta i}(\tilde{S}_{i\zeta}^{(r;k)}) - D_{\zeta i}(S_{i\zeta}^{(r;k)}) = \tilde{D}_{\zeta i}(A) - D_{\zeta i}(0) = \tilde{D}_{\zeta i}(A) > 0$. For retailer

$j \in C_{-i}(r;k)$, the difference between $\tilde{D}_{ji}(\tilde{S}_{ij}^{(r;k)})$ and $D_{ji}(S_{ij}^{(r;k)})$ is specified as,

$$\tilde{D}_{ji}(\tilde{S}_{ij}^{(r;k)})-D_{ji}(S_{ij}^{(r;k)}) = \prod_{\ell \in C(r;k)\backslash\{j,\zeta\}} [\alpha_{ji}^{\ell}+\hat{\alpha}_{ji}^{\ell}(S_{i\ell}^{(r;k)})^{\delta_{j\ell}^{i}}]\times[(\tilde{S}_{ij}^{(r;k)})^{\beta_j^i}(\alpha_{ji}^{\zeta}+\hat{\alpha}_{ji}^{\zeta}A^{\delta_{j\zeta}^{i}})-(S_{ij}^{(r;k)})^{\beta_j^i}\alpha_{ji}^{\zeta}],$$

where $\tilde{S}_{ij}^{(r;k)} = S_{ij}^{(r;k)} - A$. Similar to the proof of Theorem 1, we can show that there exists a number z_1 such that, if $\tilde{S}_{ij}^{(r;k)} > z_1$ (which means that A is a sufficiently-small positive number), then $(\tilde{S}_{ij}^{(r;k)})^{\beta_j^i}(\alpha_{ji}^{\zeta} + \hat{\alpha}_{ji}^{\zeta}A^{\delta_{j\zeta}^{i}}) > (S_{ij}^{(r;k)})^{\beta_j^i}\alpha_{ji}^{\zeta}$ and thus, $\tilde{D}_{ji}(\tilde{S}_{ij}^{(r;k)}) > D_{ji}(S_{ij}^{(r;k)})$.

For retailer $\kappa \in C_{-i}(r;k)\backslash\{j,\zeta\}$, we find that

$$\begin{aligned}\tilde{D}_{\kappa i}(\tilde{S}_{i\kappa}^{(r;k)}) - D_{\kappa i}(S_{i\kappa}^{(r;k)}) \;=\; & (S_{i\kappa}^{(r;k)})^{\beta_{\kappa}^{i}} \times \prod_{\ell \in C(r;k)\backslash\{\kappa,j,\zeta\}} [\alpha_{\kappa i}^{\ell} + \hat{\alpha}_{\kappa i}^{\ell}(\tilde{S}_{i\ell}^{(r;k)})^{\delta_{\kappa\ell}^{i}}] \\ & \times\{\alpha_{\kappa i}^{j}\hat{\alpha}_{\kappa i}^{\zeta}A^{\delta_{\kappa\zeta}^{i}} + \hat{\alpha}_{\kappa i}^{j}[(\tilde{S}_{ij}^{(r;k)})^{\delta_{\kappa j}^{i}}(\alpha_{\kappa i}^{\zeta} + \hat{\alpha}_{\kappa i}^{\zeta}A^{\delta_{\kappa\zeta}^{i}}) - (S_{ij}^{(r;k)})^{\delta_{\kappa j}^{i}}\alpha_{\kappa i}^{\zeta}]\},\end{aligned}$$

which is positive, i.e., $\tilde{D}_{\kappa i}(\tilde{S}_{i\kappa}^{(r;k)}) > D_{\kappa i}(S_{i\kappa}^{(r;k)})$, because, similar to our above proof for the result that $\tilde{D}_{ji}(\tilde{S}_{ij}^{(r;k)}) > D_{ji}(S_{ij}^{(r;k)})$, we can easily find that $(\tilde{S}_{ij}^{(r;k)})^{\delta_{\kappa j}^{i}}(\alpha_{\kappa i}^{\zeta} + \hat{\alpha}_{\kappa i}^{\zeta}A^{\delta_{\kappa\zeta}^{i}}) > (S_{ij}^{(r;k)})^{\delta_{\kappa j}^{i}}\alpha_{\kappa i}^{\zeta}$.

Therefore, we find that $\tilde{\pi}_{-i}^{(r;k)} > \pi_{-i}^{(r;k)}$, which is contrary to our above assumption that $\exists\zeta \in C_{-i}(r;k)$: $S_{i\zeta}^{(r;k)*} = 0$. It follows that the constraint $\sum_{j\in C(r;k)} \Delta\pi_j \geq 0$ is redundant and each retailer must allocate a positive space to any other retailer. Moreover, similar to the proof of Theorem 1, we can conclude that Nash equilibrium for the non-cooperative game must be unique. This theorem is thus proved. ■

Proof of Theorem 5. In order to prove that a cooperative game is essential, we must show that $\sum_{i\in N} v(C(i;1)) < v(N)$ in which $N \equiv \{1, 2, \ldots, n\}$ and $n \geq 3$. As shown in Theorem 4, for any coalition, each retailer does not secure all of his host space but allocates a part to each of other retailers. This means that all retailers in the grand coalition must operate at retailer i's store. That is, $S_{ij}^{N} > 0$, for $j \in N\backslash\{i\}$. However, when we calculate $\sum_{i\in N} v(C(i;1))$, we do not consider the excess spaces allocated by retailer i to other $(n-1)$ retailers. This means that the characteristic value $\sum_{i\in N} v(C(i;1))$ corresponds to the solutions $S_{ij} = 0$, for $j \in N\backslash\{i\}$, which are *not* optimal, as shown in Theorem 4; thus, $v(N) > \sum_{i\in N} v(C(i;1)) = 0$, which means that the game is essential.

Next, we prove that the game could be superadditive. In order to show this, we need to prove that $v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$ for any two disjoint, nonempty coalitions C_1 and C_2 such that $C_i \neq \varnothing$ $(i = 1, 2)$ ["nonempty"] and $C_1 \cap C_2 = \varnothing$ ["disjoint"]. To simplify our proof, we next consider the $k-$retailer coalition $C(r;k)$ and use general notations C_1 and C_2 to represent two space-exchange coalitions. Note that even though all retailers in the coalition $C(r;k)$ may form more than two disjoint coalitions, we can still arrive to this theorem if $v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$ for any two disjoint, nonempty coalitions C_i $(i = 1, 2)$ such that $C_i \neq \varnothing$ and $C_1 \cap C_2 = \varnothing$. For example, if, for three disjoint, nonempty coalitions $C_i \neq \varnothing$ $(i = 1, 2, 3)$, we have $v(C_1 \cup C_2 \cup C_3) \geq v(C_1 \cup C_2) + v(C_3)$ and $v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$, then $v(C_1 \cup C_2 \cup C_3) \geq v(C_1) + v(C_2) + v(C_3)$.

For this proof, we need to consider the following two cases:

1. **Both C_1 and C_2 are single-retailer coalitions and $C_1 \cup C_2$ is a two-retailer coalition**. As shown in Theorem 1, after two retailers secure their host spaces S_{ii}^N ($i = 1, 2$) and exchange their excess retail spaces, the system-wide expected profit must be no smaller than that before space exchange. This means that $v(C_1 \cup C_2) = \Delta\pi_1 + \Delta\pi_2 \geq v(C_1) + v(C_2) = 0$.
2. **Both C_1 and C_2 are nonempty coalitions, and at least one of C_1 and C_2 is a more–than-one-retailer coalition.** For this case, $C_1 \cup C_2$ is the $k-$retailer coalition $C(r; k)$ with $k \geq 3$; that is, $C_1 \cup C_2 = C(r; k)$. We find from Theorem 4 that, when the two coalitions C_1 and C_2 merge, all retailers share their spaces one another, and enjoy a positive profit surplus. When we calculate $v(C_1) + v(C_2)$, we do not consider the space that retailer i allocates to the retailers in the coalition C_2. This means that the characteristic value $v(C_1) + v(C_2)$ corresponds to the solutions ($S_{ij}^{(r;k)}$, $i, j \in C(r; k)$) such that $S_{ij}^{(r;k)} > 0$ when retailer j is in the coalition C_1, and $S_{ij}^{(r;k)} = 0$ when retailer j is in the coalition C_2. Such a solution is *not* optimal; thus, $v(C_1 \cup C_2)$ [which equals $v(C(r; k))$] must be greater than $v(C_1) + v(C_2)$, i.e., $v(C_1 \cup C_2) > v(C_1) + v(C_2)$.

This proves the theorem. ■

Proof of Theorem 6. This follows Theorem 5. More specifically, we assume that all retailers form z ($z \geq 2$) disjoint, less-than-n-retailer but nonempty coalitions $C_1, C_2, \ldots, C_z$; that is, $C_i \neq \varnothing$ and $C_i \subset C(n)$, for $i = 1, 2, \ldots, z$; $C_i \cap C_j = \varnothing$, for $i, j = 1, 2, \ldots, z$, $i \neq j$; and $\cup_{i=1}^{z} C_i = C(n)$. Thus, the total profit surpluses achieved by all retailers in the coalitions C_i ($i = 1, 2, \ldots, z$) is $\sum_{i=1}^{z} v(C_i)$, which is no more than $v(C(n))$ according to Theorem 5. This proves the theorem. ■

Proof of Theorem 7. We can solve the constrained non-linear problem to find $\hat{\mathbf{w}}$ which must exist according to Hamlen *et al.* [19], and which is unique because Γ is a convex set. Therefore, the weighted Shapley value $\hat{y}$ in terms of $\hat{\mathbf{w}}$ must ensure the stability of the grand coalition (because the resulting weighted Shapley value is in the core) and reflects the bargaining powers of n retailers (because changing the bargaining powers yields a different set $\hat{\mathbf{w}}$ and a different weighted Shapley value). ■

Appendix B The Calculation of a Unique Weighted Shapley Value for Example 2

For Example 2 in Section 4.2.4, we need to use our method in Theorem 7 to find a unique weighted Shapley value that satisfies the core conditions and also reflects the bargaining powers of three retailers. That is, given a weight set $\mathbf{w}$, we can find the corresponding weighted Shapley value as

$$\begin{aligned} y_1 &= w_1(12) \times v(12) + w_1(13) \times v(13) + w_1(123) \times [v(123) - v(12) - v(13) - v(23)], \\ y_2 &= w_2(12) \times v(12) + w_2(23) \times v(23) - w_2(123) \times \times [v(123) - v(12) - v(13) - v(23)], \\ y_3 &= w_3(13) \times v(13) + w_3(23) \times v(23) - w_3(123) \times \times [v(123) - v(12) - v(13) - v(23)]. \end{aligned}$$

Then, we should solve the following constrained nonlinear problem to find the weight set $\hat{\mathbf{w}}$.

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1,2} \left[w_i(12) - \dfrac{\gamma_i}{\gamma_1 + \gamma_2} \right]^2 + \textstyle\sum_{i=1,3} \left[w_i(13) - \dfrac{\gamma_i}{\gamma_1 + \gamma_3} \right]^2 \\ & + \textstyle\sum_{i=2,3} \left[w_i(23) - \dfrac{\gamma_i}{\gamma_2 + \gamma_3} \right]^2 + + \textstyle\sum_{i=1,2,3} \left[w_i(123) - \dfrac{\gamma_i}{\gamma_1 + \gamma_2 + \gamma_3} \right]^2 \end{aligned}$$

subject to

$$\begin{aligned} & y_1 \geq v(1) = 0, \quad y_2 \geq v(2) = 0, \quad y_3 \geq v(3) = 0; \\ & y_1 + y_2 \geq v(12), \quad y_1 + y_3 \geq v(13), \quad y_2 + y_3 \geq v(23); \\ & \textstyle\sum_{i=1,2} w_i(12) = 1, \quad \sum_{i=1,3} w_i(13) = 1, \quad \sum_{i=2,3} w_i(23) = 1, \quad \sum_{i=1,2,3} w_i(123) = 1. \end{aligned}$$

The weight set $\hat{\mathbf{w}}$ is obtained as

$$\begin{aligned} \hat{w}_1(12) &= 0.371, \quad \hat{w}_2(12) = 0.629; \quad \hat{w}_1(13) = 0.426, \quad \hat{w}_3(13) = 0.574; \\ \hat{w}_2(23) &= 0.556, \quad \hat{w}_3(23) = 0.444; \quad \hat{w}_1(123) = 0.256, \quad \hat{w}_2(123) = 0.413, \quad \hat{w}_3(123) = 0.331. \end{aligned}$$

We thus find the weighted Shapley value as

$$\begin{aligned} \hat{y}_1 &= w_1(12) \times 1129.96 + w_1(13) \times 646.46 - w_1(123) \times 1454.53 = 322.25, \\ \hat{y}_2 &= \hat{w}_2(12) \times 1129.96 + \hat{w}_2(23) \times 3317.48 - \hat{w}_2(123) \times 1454.53 = 1954.54, \\ \hat{y}_3 &= \hat{w}_3(13) \times 646.46 + \hat{w}_3(23) \times 3317.48 - \hat{w}_3(123) \times 1454.53 = 1362.58, \end{aligned}$$

which satisfies the core conditions and also reflects the bargaining powers of three retailers.