6-29-2020

Risk invulnerability : the impact of a desirable background risk

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LINGNAN UNIVERSITY

2020

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A thesis
submitted in partial fulfilment
of the requirements for the Degree of
Mater of Philosophy in Business

Lingnan University

2020

ABSTRACT

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Desirable Background Risk

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I extend the classical risk vulnerability definition proposed by Gollier and Pratt (1996) and suggest a new definition namely risk invulnerability, which is to say a desirable background risk that has a positive mean value exceeding the precautionary saving premium makes a decision maker less risk averse with respect to other independent risk. While the value function used in Milne and Robertson (1996) is comparable to the von Neumann Morgenstern utility function used in risk invulnerability, I follow the literature and show that a corporate under stochastic wealth and threat of liquidation is risk invulnerable when the wealth level of this company does not meet the dividend payment threshold. In light of this general case, I propose a specific application of an insurance company suggested by Rochet and Villeneuve (2011), which is facing a zero-or-full reinsurance strategy because of a huge risk. I first confirm that a non-dividend paying insurance company is risk invulnerable, and then investigate the effect of reinsurance on risk invulnerability. I propose that, when the insurance company switch its reinsurance strategy from zero to full coverage, there is an instant decrease in the level of risk invulnerability reflected by the change of magnitude of the corresponding conditions.
Key words: Risk invulnerability, desirable risk, increasing absolute prudence, intemperance, corporate value

DECLARATION

I declare that this is an original work based primarily on my own research, and I warrant that all citations of previous research, published or unpublished, have been duly acknowledged.

SIGNED

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CERTIFICATE OF APPROVAL OF THESIS

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ACKNOWLEDGEMENT

First, I would like to express my gratitude to my supervisors Prof. Li Jingyuan for his patience, knowledge and continuous support of my MPhil study. He also inspired me to continue in the academia by combining theoretical and empirical researches. Besides, I would like to thank the rest of my thesis committee for their valuable and inspirational advice.

I would also extend my appreciation to Prof. Keith K. P. Wong, Prof. Ho Yin Yick and Dr Wang Jianli for their guidance and suggestions during the preparation of the thesis. Furthermore, I want to thank Prof. Cheng Jiang for attending my seminar, and Ms. Clara Hui and Ms. Agnes Tsui for their helps during the years of study.

Last but not least, I would like to thank my fellow classmates for the communications and discussions which motivated me and cleared my vision.

1 Introduction

As an investor, he may faces some background risks other than the investment risks generated by holding risky assets, such as labor income (i.e. future income uncertainty), housing risks (i.e. fluctuation of housing market or rent) etc.. Classical studies on background risk suggest that its existence increases an individual's risk aversion towards other independent risks. While this may not be the absolute case, Gollier and Pratt (1996) propose the concept of risk vulnerability and investigate that, under which conditions, an unfair background risk guarantees that a decision maker (DM) behaves in a more risk averse way. Classic papers by Gollier and Pratt (1996), Eeckhoudt et al. (1996), Franke et al. (2006), and others explore conditions for risk vulnerability, as well as its implications. In contrast to this result, I examine the effect of a desirable background risk on the attitude of a DM to bear other independent risks.

I first extend the classical risk vulnerability concept proposed by Gollier and Pratt (1996) by showing that a DM can still be risk vulnerable under desirable background risks. Then I introduce a concept of risk invulnerability, namely that, imposing a desirable background risk under certain condition reduces the DM's risk aversion.

While most of the existing literature about risk preferences are mainly at individual level, I bring the analysis to a corporate level and examine the corporate risk preferences determined by managerial decision makers. Conventionally understanding the firm behaviors depends on the decision makers' (managers) preference. However, there is no agency problem in our corporate setting same as Milne and Robertson (1996), meaning that managers make decisions at the best to the shareholders. The primary objective of managers are to maximize the shareholders' value, or equivalently firm value function, which is actually the present value of the shareholders' expected utility of the managerial decisions. Therefore, managerial risk preference can be ignored and only shareholders' should be focused on. Given by this stand point, I question whether some individual risk behaviors or preferences can be applied on companies since they are founded and controlled by a group of individuals. In order to answer this question, the function that can represent every single shareholder should be used, which is the firm value function. Besides, according to Sitkin and Pablo (1992), the factors that alter firm's risk behaviors and preferences include characteristics of the company. It is difficult to conclude the results or control the related firm characteristics if I simply apply the traditional individual theory to each shareholders of the company.

Furthermore, since multiple existing literature conduct analysis by assuming shareholders are risk neutral (Milgrom and Roberts, 1992; Wright et al., 2002; Sanders and Hambrick, 2007), or equivalently the firms have linear risk preference (Nelson, 1961; Tisdell, 1963, Dreze and Gabszewicz, 1967; Mills 1961), I incorporate some firm level background risk (e.g. human capital risk, operational risk etc.) and other independent threats to investigate the risk behaviors and preferences of risk averse shareholders (firms). To be specific, I would like to examine the risk vulnerability and invulnerability theory to firms with both background risk and other independent threats and see whether the theories are applicable.

With the above inspiration, I then apply my result obtained at individual analysis to a continuous time framework in which a firm has a wealth following stochastic process and faces a threat of liquidation as it is not allowed to perform external financing. Given that the shareholders and the firm are risk averse and prudent to the threat of liquidation, I further identify that my proposed conditions for risk invulnerability hold when background risk is small and so that the firm is risk invulnerable to the threat. This corporate level application thus creates useful implications to explain multiple corporate behaviors. I also provide a specific application of my theory on an insurance company setting suggested by Rochet and Villeneuve (2011). Such kind of firms are then shown to be risk invulnerable and investigated how this nature changes in accordance with the reinsurance strategy.

The thesis is organized as follows. In Chapter 2, I briefly introduce the developments of the existing literature. In Chapter 3, I propose the definition of risk invulnerability and provide a sufficient condition for it. Then, a corporate level application will be included in Chapter 4. I conclude the whole thesis in the last chapter.

2 Literature Review

2.1 Developments of Risk Aversion and Prudence

Since the coefficient of risk aversion has first been approximated by Pratt (1964), it has been extended to different theories and applied in different fields.

Pratt (1964) suggest that the level of absolute risk aversion can be approximated by a risk premium, the amount that a decision maker is willing to pay to reduce the risk to its expectation. The coefficient measures how the concavity of a von Neumann–Morgenstern utility function changes.

Then a similar technique has been adopted if the risk happened in the future, like Kimball (1990) who develops a concept of prudence. It is referred by the precautionary saving motive induced by the future risk. The level of prudence is then approximated by the precautionary premium, which is similar to the concept of risk premium.

Pratt and Zeckhauser (1987) make reference with the development of decreasing absolute risk aversion (DARA) which demonstrates that the level of risk aversion, measured by risk premium, decreasing with respect to wealth and propose a similar behavior called proper risk aversion. This suggests that a decreasing risk averse DM will never find a undesirable lotteries desirable due to an introduction of other undesirable lotteries.

Similarly, Kimball (1993) utilizes these concepts and identified a behavior called standard risk aversion, which incorporated both DARA and decreasing absolute prudence (DAP). The concept utilizes the measurement of prudence suggested by Kimball (1990). DAP can then be interpreted as a decreasing relationship between precautionary saving motive or precautionary premium and wealth. Standard risk aversion refers to a certain type of decision maker who did not prefer "all or nothing", which means he prefers to separate two risks and put them in different states rather than bearing both in the same state. Standard risk aversion is a weaker concept because, while the proper risk aversion requires two undesirable risks, standard risk aversion only requires one undesirable risk and one loss-aggravating risk.

Gollier and Pratt (1996) introduce a concept called risk vulnerability. It is defined as a DM will be more risk averse to other independent risk due to an introduction of an unfair background risk. Decreasing and concave absolute risk aversion is the necessary and sufficient conditions of risk vulnerability. Gollier and Pratt (1996) also demonstrate that risk vulnerability is the weakest concept of the above by proving that all proper and standard utility function have to be risk vulnerable.

2.2 Developments of Corporate Liquidity Management

Among the first to utilize liquidity management models in corporate level analysis, such as Jeanblanc-Picque and Shiryaev (1995), Radner and Shepp (1996) and Milne and Robertson (1996) etc., general type of corporations are used in the literature.

Radner and Shepp (1996) try to identify the optimal strategy of the production technology to maximize the firm value and, also, identify that, since the company will choose a greater volatility/drift ratio once the wealth holding increases, they propose that every company is going to liquid in a finite time period.

Milne and Robertson (1996) examine a case where the company has to determine the optimal dividend strategy to maximize the company's value. They identify that a corporate facing a threat of liquidation and having insufficient wealth to meet the threshold of paying dividends has the nature of risk aversion and DARA.

One stream of the existing literature intend to smooth or add some boundaries to produce interesting results. For example, normally, in order to create a framework that management liquid reserve is necessary to the company, some boundaries such as no external financing, liquidation when company's wealth drops to zero etc., are imposed. Specifically,

Radner and Shepp (1996) smooth the single production technology problem in Jeanblanc-Picque and Shiryaev (1995) by creating a multi-technology framework and Jiang and Pistorius (2012) impose a "regime switches" which allows company not to be bounded by one regime only and shows how the dividend paying threshold changes in accordance with the regime adopted by the company.

Paulsen and Gjessing (1997) smooth the setting of "save to survive" and allow investment by using the retained earnings. Also, instead of incorporating Brownian risk, they also allow a compound Poisson process to be one of the random driver of the company's net earnings.

Furthermore, while Stokey (2009) allows equity issuance and add a financial friction which is a fixed issuance cost to the model, Scheer and Schmidli (2011) follow a similar framework but incorporate a jump in the stock price dynamic model; Lokka and Zervos (2008) smooth the assumption of fixed issuance cost and propose proportional issuance cost. Literature which smooths certain conditions is intended to polish the models in a more realistic way.

Besides the general application, several literature have brought the models to different industries like insurance industry. Traditionally, liquidity management model in insurance company are used in minimizing the probability of ruin, which means the possibility of liquidation due to poor underwriting and high risk exposure of the company.

After De Finetti (1957), saying that the insurance company should have an alternative objective which is firm value maximization, the Cramer-Lundberg model has been well suggested to model an insurance company which is facing the dividend distribution problem under not only a Brownian risk but also a huge loss following a process of compound Poisson process.

Hoejgaard and Taksar (1998) suggest that the insurance firm facing the Brownian risk only have a possibility to seek reinsurance. The firm has to decide the optimal reinsurance strategy to control the risk exposure and maximize the firm value. The study aims at identifying what is the optimal proportional reinsurance coverage and how this proportion affects the dividend distribution threshold.

Furthermore, Rochet and Villeneuve (2011) combine the Cramer-Lundberg model and Hoejgaard and Taksar (1998) and impose an optimization problem of the insurance firm under both two types of risks.

Eisenberg and Schmidli (2011) adopt a similar framework but, instead of identifying an optimal reinsurance strategy to maximize firm value, they focus on minimizing the probability of ruin.

2.3 Contributions of the Thesis

My thesis provides new conditions to the traditional risk vulnerability theory, proposes the sufficient conditions for the alternative theory, namely risk invulnerability, and examines this theory in two separate corporate cases.

Firstly, tradition economic theories (Pratt and Zeckhauser, 1987; Kimball , 1993; Gollier and Pratt 1996) mainly focus on analyzing behavior under undesirable risks. They seldom study desirable risks. I show the intuition that any desirable background risks will decrease the DM's intensity of risk aversion towards other independent risk is not flawless. Instead, I demonstrate several conditions that prove a desirable background risk which have an expected value lower than the precautionary premium can still put the DM in a vulnerable position. In other words, my study releases the condition of risk vulnerability.

Secondly, I suggest an under-explored combination of condition which decreasing absolute risk aversion (DARA) and increasing absolute prudence (IAP) to be the sufficient conditions of risk invulnerability.

Thirdly, I use firm value function to represent the expected utility of a group of shareholders reacting to the managerial decisions as a whole and conclude that individual risk behaviors and preferences due to a background risk and other independent threats are applicable in firm level. I incorporate a general corporate case suggested by Milne and Robertson (1996) and a specific insurance company case proposed by Rochet and Villeneuve (2011), which may lead to a series of investigations on applying the findings from individual behavioral theories on firms, or lead the analysis to identify different firm level risk taking behaviors corresponding to the preferences.

Unlike Gollier and Pratt (1996) which examine risk vulnerability only, my study propose the determinants of the intensity of risk invulnerability and identify that switching from full-coverage to zero-coverage reinsurance strategy instantly decreases the intensity of risk invulnerability by showing greater DARA and IAP.

3 Model

3.1 Characterizations of Risk Vulnerability and Invulnerability

Consider a DM whose preference for wealth w can be represented by an expected utility. I let $u(.)$ denote the utility function. In this thesis, I assume any utility function $u(w)$ in the following is $u' > 0$, $u'' \leq 0$ and $u''' \geq 0$. $\widetilde{x}$ is a background risk. Kimball (1990) proposes the following measurement of precautionary saving premium.

Definition 3.1. *(Kimball 1990) The precautionary saving premium $\psi(\widetilde{x}, w)$ is defined by*

$$Eu'(w + \widetilde{x}) = u'(w + E\widetilde{x} - \psi). \tag{1}$$

The precautionary saving premium directly measures the money that the DM is willing to pay to remove the effect of the risk on the marginal utility function.

Gollier and Pratt (1996) propose the following definition:

Definition 3.2. *(Gollier and Pratt 1996) Preference function u displays "risk vulnerability" if the presence of an exogenous background risk with a nonpositive mean, namely, an unfair risk, raises the aversion to any other independent risks, or equivalently,*

$$E\widetilde{x} \leq 0 \Rightarrow -\frac{Eu''(w + \widetilde{x})}{Eu'(w + \widetilde{x})} \geq R(w)\ \forall w. \tag{2}$$

where $R(.) = -\frac{u''(.)}{u'(.)}$ is the absolute risk aversion coefficient. Risk vulnerability means that the DM is more risk averse when she is exposed to such a background risk than when she is not.

I propose the following definition to extend the classic concept of risk vulnerability.

Definition 3.3. *Preference function u can exhibit "risk vulnerability" in the presence of an exogenous background risk if it increases the aversion to any other independent risks.*

The above definition does not require the background risk has a non-positive mean value. In light of this definition, I propose the following sufficient condition.

Proposition 3.4. *The presence of a background risk increases the aversion to other risks; that is to say, the utility is risk vulnerable, if the following three conditions are fulfilled:*

1. *DARA;*
2. *DAP;*
3. $E\widetilde{x} \leq \psi \ \forall w$.

Proof. See Appendix A. □

The above proposition provides evidence that a desirable risk can still make a DM becomes more risk averse, which says that she is still vulnerable to other independent risks because the background project does not yield an expected value high enough.

Now I examine the impact of a background risk that has a high enough in expected value. The intuition suggests that such a background risk must decrease the aversion to other independent risks. It should induce the DM to buy less insurance, and to increase her demand for a risky asset. To investigate this intuition, I propose the following definition:

Definition 3.5. *Preference function u exhibit "risk invulnerability" if the presence of an exogenous positive mean value background risk reduces the aversion to any other independent risks.*

Then, I provide a sufficient condition for risk invulnerability in the following proposition.

Proposition 3.6. *The presence of a risk reduces the aversion to other risk; that is to say, the utility is risk invulnerable, if the following three conditions are satisfied:*

1. *DARA;*
2. *IAP;*
3. $E\widetilde{x} \geq \psi \ \forall w$.

Proof. See Appendix B. □

Proposition 3.6 shows that, in order to guarantee that the introduction of a background risk reduces the aversion to other independent risks, I need not only DARA and IAP, but also a sufficiently high mean value of the background risk. A DM is risk invulnerable to any other risks if he has DARA and IAP in nature and the background project earns high enough to fulfil his precautionary saving premium.

3.2 Intensity of Risk Invulnerability

From proposition 3.6, by fulfilling the three conditions, the DM who is facing a desirable background risk high enough to cover the precautionary saving premium will be less risk averse to other independent risk. I hereby question how the intensity of risk invulnerability, measured by the level of risk aversion reduced by the introduction of a desirable background risk, is affected in response to the changes of the conditions.

To calculate the level of risk aversion reduced, I denote $-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})}$ as $\Re(w)$ and calculate

$$
\begin{aligned}
\frac{\Re(w)-R(w)}{R(w)} &= R(w)^{-1}\Big[-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})}-R(w)\Big] \\
&= R(w)^{-1}\Big[-\frac{E[u''(w)+\widetilde{x}u'''(w)+0.5\widetilde{x}^2u^{(4)}(w)]}{E[u'(w)+\widetilde{x}u''(w)+0.5\widetilde{x}^2u'''(w)]}-R(w)\Big] \\
&= \frac{1-E\widetilde{x}P(w)+0.5E\widetilde{x}^2P(w)T(w)}{1-E\widetilde{x}R(w)+0.5E\widetilde{x}^2P(w)R(w)}-1 \\
&= \frac{E\widetilde{x}[R(w)-P(w)]+0.5E\widetilde{x}^2[P(w)T(w)-P(w)R(w)]}{1-R(w)[E\widetilde{x}-0.5E\widetilde{x}^2P(w)]} \\
&= \frac{[E\widetilde{x}-0.5E\widetilde{x}^2P(w)][R(w)-P(w)]+0.5E\widetilde{x}^2P(w)[T(w)-P(w)]}{1-R(w)[E\widetilde{x}-0.5E\widetilde{x}^2P(w)]}
\end{aligned}
$$

where $P(w)=-\frac{u'''(w)}{u''(w)}$ is the level of prudence; $T(w)=-\frac{u^{(4)}(w)}{u'''(w)}$ is the level of temperance. If the risk $\widetilde{x}$ is infinitesimally small and positive, $\frac{\Re(w)-R(w)}{R(w)}$ should be less than zero if DARA ($P(w)\geq R(w)$) and IAP ($P(w)\geq T(w)$) holds. This calculation not only confirms the conditions included in proposition 3.6, but also shows that the level of risk aversion reduced by the presence of desirable background risk, or equivalently the intensity of risk invulnerability, is dependent on how much the conditions change.

In here, $R(w)-P(w)$ and $T(w)-P(w)$ are only one of the methods to measure how much the level of risk aversion and prudence change. In Chapter 4 I will use the first derivative of the risk aversion's and prudence's coefficient to measure directly the change of these two attitudes.

3.3 Measuring Small Desirable Risk

Now consider a situation where the size of the desirable background risk is infinitesimally small. I provide the following proposition:

Lemma 3.7. *$E\widetilde{x}\geq\psi$ holds when the positive mean value background risk is infinitesimally small.*

Proof. See Appendix C. □

The purpose of discussing this small-risk situation is that, as soon as this proposition holds, I can examine the risk invulnerability in continuous framework. If a company instead of an individual is facing infinite small background risks, the wealth should follows the stochastic process. Also, since I show that $E\widetilde{x} \geq \psi$ holds, I only have to identify the nature of risk aversion and prudence of a company. This proof becomes a bridge between individual discrete analysis and company analysis.

4 Corporate Risk Invulnerability

4.1 Corporations under Brownian Risk

In this part, I consider here an example applying Proposition 3.6 and Lemma 3.7. In particular, since $E\widetilde{x} \geq \psi$ holds when $\widetilde{x}$ is small, I would like to demonstrate that DARA and IAP are naturally implied by the firms mentioned below.

Milne and Robertson (1996) consider that the movement of a corporation's wealth follows a stochastic process m_t and the corporation has to decide when to pay dividends l_t to maximize corporation value under the threat of liquidation, because the firm can not perform external financing. It can only save money to escape from liquidation. In this case, the firm is exposed to both background risk from the stochastic process of wealth and other independent risk from the threat of liquidation. The movement of m_t is defined as follow:

$$dm_t = (rm_t + \mu - l_t)dt + \sigma dZ, \qquad m_0 = w, \tag{3}$$

where r is the reward of saving; μ is the drift term; σ is the volatility and Z is a Brownian motion.

The value of the shareholders and the corporation, denoted as V, is an expected utility of the discounted stream of future optimal dividends determined by the managers.

$$V(w) = E[\int_0^\infty e^{-\rho\tau} u(l_\tau) d\tau | m_0 = w], \tag{4}$$

where $\rho > r$ is the shareholders' discount rate. Using the Hamilton-Jacobi-Bellman equation, Milne and Robertson (1996) determine the optimal control flow (the dividend flow) in different the states (cash reserves) to achieve the objective, which is maximizing the value function, as follow

$$\rho V = \max_{l_t} \left\{ u(l) + (rm_t + \mu - l_t)V' + \frac{1}{2}\sigma^2 V'' \right\} \tag{5}$$

Then they impose some boundary conditions to this value function to identify some of the characteristics. For example, The conditions are (1) corporation will liquidate if $V = 0$; (2) $V' = 1$ and $V'' = 0$ at m^*, where m^* is the dividend payment threshold for wealth; (3) for $m_t \geq m^*$, $V(m_t) = V(m^*) + m_t - m^*$ and (4) $V' \geq 1$, $V'' \leq 0$ and $V''' > 0$ within $0 < m < m^*$. Notice that the value function of the company mentioned is not only representing a groups of shareholders' expected utility function, but also incorporate some firm specific characteristics and boundaries, which helps us to obtain a thorough understanding on firm's risk preferences. If saving is not rewarded (i.e. $r = 0$) and there is no dividend payout, the company will only save money and has the following equation:

$$\frac{1}{2}\sigma^2 V'' + \mu V' - \rho V = 0, \quad 0 < m < m^*, \tag{6}$$

and a co-state equation

$$\frac{1}{2}\sigma^2 V''' + \mu V'' - (\rho - r)V' = 0. \tag{7}$$

Based on these two equations, I propose that

Proposition 4.1. *Over the non-dividend paying region $0 < m < m^*$, a corporation under Brownian risk and threat of liquidation has DARA and IAP, and therefore is risk invulnerable.*

Proof. See Appendix D. □

The corporation is concluded to be risk invulnerable given by its stochastic process, DARA, IAP in nature, meaning that the shareholders' and the companies' willingness to bear other risks is invulnerable due to the introduction of a desirable background risk only when its expected value exceeds the precautionary saving premium. Here the corporation's risk invulnerability is endogenous and stems from the financial frictions: even though shareholders are risk neutral or fully diversified, the corporation behaves in a risk-invulnerable way because a desirable background risk may reduce the threat of liquidation.

4.2 Insurance Company under Huge Loss and Reinsurance

With reference to the above findings, I extend the previous application to an insurance company facing, Brownian risk, huge loss and reinsurance opportunity. I demonstrate that the conditions of risk invulnerability still hold in this specific application. In Rochet and Villeneuve (2011), an insurance company has the following dynamics of liquid reserves under a background risk, independent huge loss and reinsurance opportunity:

$$dM_t^\pi = (\mu - i_t l\lambda)dt + \sigma dZ_t - dL_t - (1 - i_t)ldN_t \tag{8}$$

where $\mu > i_t l\lambda$ and there is no external financing; $i = \{i_t \in [0, 1], t \geq 0\}$ is proportional reinsurance process, measuring the fraction of l which is re-insured; l is a large devastating loss which makes $V(m-l) = V(0) = 0$; $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity λ; the probability that a huge loss happens over an interval $[t, t + dt]$ is λdt; L is a cumulative dividend process.

Same as the last application, the insurance company is exposed to liquidation threat so that it has to determine its optimal dividend payment and proportion of reinsurance. show that the dividend threshold, denoted as m_1^*, exists in which $V' = 1$ and $V'' = 0$. Within $[0, m_1^*]$, while there is no dividend payment, the HJB equation becomes

$$(\rho + \lambda)V(m) = \frac{\sigma^2}{2}V''(m) + \max_{i\in[0,1]}\{(\mu - i_t l\lambda)V'(m) + \lambda V(m - (1 - i)l)\}. \tag{9}$$

To decide the optimal reinsurance strategy, the first-order condition of the following

$$\max_{i\in[0,1]}\{(\mu - i_t l\lambda)V'(m) + \lambda V(m-(1-i)l)\} \tag{10}$$

with respect to i yields

$$\lambda l[V'(m-(1-i)l) - V'(m)] = 0. \tag{11}$$

Given that $V'(m)$ is decreasing, the only solution for $m-(1-i)l = m$ is $i = 1$. It means that there is a corner solution at a threshold m_0^* in which the insurance company will switch from zero to full reinsurance. While it is obvious that condition (11) is larger than zero if $i = 0$, the optimal reinsurance strategy is

$$\begin{aligned} i^* &= 0 \quad if \quad m \le m_0^* \\ &= 1 \quad if \quad m \in [m_0^*, m_1^*] \end{aligned}$$

In the full reinsurance situation, on $m \in [m_0^*, m_1^*]$ where $i^* = 1$, the state equation and co-state equation become

$$\frac{\sigma^2}{2}V''(m) + (\mu - l\lambda)V'(m) - \rho V(m) = 0 \tag{12}$$

and

$$\frac{\sigma^2}{2}V'''(m) + (\mu - l\lambda)V''(m) - (\rho - r)V'(m) = 0, \tag{13}$$

where $\mu'(m) = r$. With regards to these two equations, I propose that:

Proposition 4.2. *Under the zero-exposure region, the insurance company is risk invulnerable.*

Proof. See Appendix E. □

I observe that, under full reinsurance coverage, the insurance company has the same level of DARA and IAP as the company mentioned by Milne and Robertson (1996). Since the huge loss is fully reinsured, the insurance company has to bear the Brownian risk like a general corporate, and therefore, they are similarly risk invulnerable.

In contrast, on the region $m \le m_0^*$ where $i^* = 0$, I first propose that an insurance company facing huge risk is risk invulnerable even there is zero reinsurance. When there is no dividend payments, I have the state equation

$$\frac{\sigma^2}{2}V''(m) + \mu V'(m) - (\rho + \lambda)V(m) = 0, \tag{14}$$

as $V(m-l) = 0$, and the co-state equation

$$\frac{\sigma^2}{2}V'''(m) + \mu V''(m) - (\rho + \lambda - r)V'(m) = 0. \tag{15}$$

With regards to these two equations, I propose that:

Proposition 4.3. *An insurance company which is entirely not covered by the reinsurance has greater intensity of risk invulnerability.*

Proof. See Appendix F. □

If I compare the results obtained from full and zero reinsurance coverage , the insurance company with zero reinsurance coverage has a greater DARA (by $\frac{2\lambda}{\sigma^2}\left[\frac{V'}{V''} - \frac{V}{V'}\right]$) and IAP (by $\frac{2\lambda}{\sigma^2}\left[\frac{(V'')^2 - V'V'''}{(V'')^2}\right]$). Given that the insurance is fully exposed to the huge loss, according to the intensity of risk invulnerability, greater DARA and IAP means the DM would be more risk averse in the presence of desirable background risk. Equivalently, the DM has greater intensity of risk invulnerability.

5 Implications of Risk Invulnerability

Under the setting with no agency problem, risk preference like DARA and IAP reflected by firm value function represents the risk preference of a group of shareholders. Individual theories like risk vulnerability and invulnerability are applicable in firm level due to desirable background risk and other independent threats.

In terms of corporate investment, overestimating the demand on risky assets when constructing the optimal portfolio may happens easily if background risk is not considered (Mehra and Prescott, 1985; Weil, 1992). Risk invulnerability ensures that introducing a simple desirable risk may not be helpful enough to the company to conclude its optimal investment on risky asset. In the case where a desirable risk earns high enough in expectation to cover the company's precautionary saving motive, optimal risky investment should be increased. In particular, risk invulnerability ensures that, if a company has this first order condition on an independent risk $E\widetilde{y}u'(w + \alpha^*\widetilde{y}) = 0$, DARA and IAP are sufficient to conclude $\alpha^* E\widetilde{y}u'(w + z + \alpha^*\widetilde{y}) \geq 0$ if the background risk has $z = E\widetilde{x} \geq \psi > 0$.

It is common to interpret that a decreasingly risk averse firm will raise its holding of risk assets if wealth increases. In addition to the change of risk premium which shows how risk aversion drops in response to the increase of wealth, IAP here may serve as another benchmark signalling the company when to be less risk averse. While the company is getting wealthier but riskier, it is preventive and conservative to create an buffer of saving increasing alongside the wealth and riskiness.

6 Conclusion and Potential Developments

In this thesis, I extended the classical risk vulnerability concept by considering a desirable background risk. A DM can still be risk vulnerable if there are DARA, DAP and $E\widetilde{x} \leq \psi$. Then I introduced a concept of risk invulnerability. It holds if a desirable background risk which has a positive expected value exceeding the precautionary saving premium makes a DM becomes less averse to other independent risks, i.e. for $E\widetilde{x} \geq \psi \Rightarrow -\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})} \leq R(w)\ \forall w$ in case where DARA and IAP hold. It can be interpreted as the background project earns significant enough to cover the target saving amount or fulfill the willingness to save.

Furthermore, I show that $E\widetilde{x} \geq \psi$ holds when the size of the desirable background risk is infinitesimally small and the level of risk aversion reduced due to the presence of desirable background risk, namely intensity of risk invulnerability, is dependent to magnitudes of DARA and IAP. Since the corporate value function is the discounted utility of a group of shareholders towards managerial decisions, we apply risk invulnerability to corporate level analysis. I first examined my theory by identifying that the nature of a corporation from Milne and Robertson (1996) under a threat of liquidation and zero dividend payments is DARA and IAP, which lead us to a conclusion that such corporate is risk invulnerable.

In light of this general application, I expand the study by covering the insurance company under the settings from Rochet and Villeneuve (2011). This insurance company is facing a huge loss with proportional reinsurance possibility. I identify that, while the company's huge loss is fully reinsured, the company has the same level of DARA and IAP, or equivalently same intensity of risk invulnerability, to the general corporate suggested by Milne and Robertson (1996). If the insurance company is entirely exposed to the huge loss, greater DARA and IAP have been identified, meaning that the insurance company is acting more preventively and precautionarily.

As the implication of IAP together with and application in the corporate level are under-explored, I expect that more corporate behaviors such as preferences over financing and capital structure, or firms with different industrial characteristics will be covered by risk invulnerability. Also, with regards to the alternative strategy other than zero- or full-reinsurance policy adopted by the insurance company, an unexplored area that this thesis can be extended to is the insurance or reinsurance company which adopts different strategies of insurance linked security. By determining some existing strategies (Niehous, 2002; Trottier, 2017) which is optimal to firm value maximization, I may investigate whether such kind of company is risk invulnerable.

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Appendix

Appendix A: Proof of Proposition 3.4

Proof. Differentiating Eq (1) with respect to w and yield

$$Eu''(w+\widetilde{x}) = (1-\psi')u''(w+E\widetilde{x}-\psi) \tag{16}$$

and

$$-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})} = (1-\psi')R(w+E\widetilde{x}-\psi). \tag{17}$$

Decreasing absolute prudence (DAP) (or say, $\psi' \leq 0$) implies

$$-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})} \geq R(w+E\widetilde{x}-\psi). \tag{18}$$

Decreasing absolute risk aversion (DARA) and $E\widetilde{x} \leq \psi$ imply

$$R(w+E\widetilde{x}-\psi) \geq R(w). \tag{19}$$

Therefore I obtain $-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})} \geq R(w)$. □

Appendix B: Proof of Proposition 3.6

Proof. Eq (17) and IAP (or say, $\psi' > 0$) imply

$$-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})} \leq R(w+E\widetilde{x}-\psi). \tag{20}$$

DARA and $E\widetilde{x} \geq \psi$ imply

$$R(w+E\widetilde{x}-\psi) \leq R(w). \tag{21}$$

Therefore I obtain $-\frac{Eu''(w+\widetilde{x})}{Eu'(w+\widetilde{x})} \leq R(w)$. □

Appendix C: Proof of Lemma 3.7

Proof. Recall equation (1), it can be re-written as:

$$Eu'(w+k\widetilde{x}) = u'(w+Ek\widetilde{x}-\psi(k)) \tag{22}$$

where $k \in [0,1]$ measures the size of the desirable background risk. By differentiating the equation with k, I obtain:

$$E\widetilde{x}u''(w+k\widetilde{x}) = [E\widetilde{x}-\psi'(k)]u''(w+Ek\widetilde{x}-\psi(k)). \tag{23}$$

When $k \to 0$, $\psi \to 0$. If $k=0$, then equation (23) becomes

$$E\widetilde{x}u''(w) = [E\widetilde{x}-\psi'(0)]u''(w). \tag{24}$$

Thus $\psi'(0) = 0$.

Taking derivative with respect to k on both sides of Eq (23), I obtain:

$$E(\widetilde{x}^2)u'''(w + k\widetilde{x}) = E\widetilde{x}[E\widetilde{x} - \psi'(k)]u'''(w + Ek\widetilde{x} - \psi(k)) - \left[\psi''(k)u''(w + Ek\widetilde{x} - \psi(k)) + \psi'(k)[E\widetilde{x} - \psi'(k)]u'''(w + Ek\widetilde{x} - \psi(k))\right]. \quad (25)$$

When $k = 0$,

$$E(\widetilde{x}^2)u'''(w) = (E\widetilde{x})^2 u'''(w) - \psi''(0)u''(w). \quad (26)$$

With some simple rearrangements, I obtain

$$\psi''(0) = \sigma_x^2 P(w) \quad (27)$$

where σ_x^2 is the variance of the desirable background risk.

Applying Taylor expansion on $\psi(k)$, I obtain

$$\psi(k) = k\psi'(0) + k^2\psi''(0) + o(k^2). \quad (28)$$

Since $\psi(0)$ and $\psi'(0)$ equal to zero, combining equation (27) and (28), I have

$$Ek\widetilde{x} - \psi(k) = Ek\widetilde{x} - k^2\sigma_x^2 P(w) \geq 0.$$

When k is infinitesimally small, k is significantly larger than k^2. I can conclude that $Ek\widetilde{x} \geq \psi(k)$ holds. □

Appendix D: Proof of Proposition 4.1

Proof. Milne and Robertson (1996) define

$$A = -\frac{V''}{V'} \quad (29)$$

and apply equation (7) to obtain that $A' < 0$ (Milne and Robertson (1996), Proposition 3). It illustrates that the corporation under the settings of Milne and Robertson naturally imply decreasing absolute risk aversion.

It can be shown that (see, Milne and Robertson (1996), (A.4))

$$\frac{A'}{A} = A + \frac{V'''}{V''} = \frac{2}{\sigma^2}\left[(\rho - r)V'/V'' - \rho V/V'\right] \quad (30)$$

Based on the fourth conditions mentioned above (Milne and Robertson (1996), Proposition 2), it is obvious that the result in equation (30) is negative. Define $P = -\frac{V'''}{V''}$, I have

$$P = -\frac{2}{\sigma^2}\left[(\rho - r)V'/V'' - \mu\right] \quad (31)$$

By differentiating P, I obtain

$$P' = -\frac{2}{\sigma^2}\left[(\rho - r)\frac{(V'')^2 - V'V'''}{(V'')^2}\right] + \frac{2r}{\sigma^2} \geq 0 \quad (32)$$

because DARA implies $(V'')^2 - V'V''' \leq 0$. Under the normal assumption that $\rho > r$, the slope of P is positive based on the settings of Milne and Robertson (1996). Therefore, the first two conditions in Proposition 3.6 are satisfied. □

Appendix E: Proof of Proposition 4.2

Proof. The coefficient of risk aversion

$$A = -\frac{2}{\sigma^2}\big[\rho\frac{V}{V'} - \mu\big] \tag{33}$$

and the coefficient of prudence

$$P = -\frac{V'''}{V''} = -\frac{2}{\sigma^2}\big[(\rho - r)\frac{V'}{V''} - \mu\big] \tag{34}$$

are derived the state and co-state equations. The change of risk aversion and prudence are

$$\frac{A'}{A} = A + \frac{V'''}{V''} = \frac{2}{\sigma^2}\big[-\rho\frac{V}{V'} + (\rho - r)\frac{V'}{V''}\big] \leq 0 \tag{35}$$

given that $V' > 0$, $V'' < 0$ and $V''' > 0$. Then, by differentiating P with respect to wealth, I obtain

$$P' = -\frac{2}{\sigma^2}(\rho - r)\big[\frac{(V'')^2 - V'V'''}{(V'')^2}\big] + \frac{2r}{\sigma^2} \geq 0 \tag{36}$$

I conclude that the first two conditions in Proposition 3.6 are satisfied. □

Appendix F: Proof of Proposition 4.3

Proof. The coefficient of risk aversion

$$A = -\frac{2}{\sigma^2}\big[(\rho + \lambda)\frac{V}{V'} - \mu\big] \tag{37}$$

is derived from (14) and the coefficient of prudence

$$P = -\frac{V'''}{V''} = -\frac{2}{\sigma^2}\big[(\rho + \lambda - r)\frac{V'}{V''} - \mu\big] \tag{38}$$

is derived from (15). Combining these results can obtain

$$\frac{A'}{A} = A + \frac{V'''}{V''} = \frac{2}{\sigma^2}\big[-(\rho + \lambda)\frac{V}{V'} + (\rho + \lambda - r)\frac{V'}{V''}\big] \leq 0, \tag{39}$$

given that $V' > 0$, $V'' < 0$ and $V''' > 0$. Then, by differentiating P with respect to wealth, I obtain

$$P' = -\frac{2}{\sigma^2}(\rho + \lambda - r)\big[\frac{(V'')^2 - V'V'''}{(V'')^2}\big] + \frac{2r}{\sigma^2} \geq 0. \tag{40}$$

I not only conclude first two conditions in Proposition 3.6 are satisfied, but also confirm that reinsurance coverage has instant effect on the intensity of risk invulnerability. □