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# A Fair Staff Allocation Rule for the Capacity Pooling of Multiple Call Centers

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We construct a cooperative staffing game to investigate how to fairly allocate the reduced number of staffs among multiple call centers that pool (centralize) their capacities. We show that this game is essential and submodular, and thereby, convex with a non-empty core. We also propose a neat Shapley value-characterized staff-allocation rule, which exists in the core of the game.

**Key words:** Call center pooling, square-root safety staffing rule, cooperative game, Shapley value.

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## 1 Introduction

The past decade has witnessed a number of successful practices in which multiple call centers centralize their capacities to achieve scale economy or other pooling effects. Such practices are particularly common to the small spatially-separated call centers, which are more willing to be virtually or physically pooled into a large call center. For example, a U.S. bank has four call centers serving customers in different regions. An incoming call from a region is first assigned to the center that is located in the region. If the call's waiting time reaches 10 seconds, then the call will be sent to an interqueue and then be answered by all four call centers, depending on which center has an available agent [6]. In fact, the telecommunication technique is highly capable of realizing the above virtual pooling system. In addition to the wide existence of the pooling of separate call centers, such a strategy has also been implemented within a call center. For example, the agents in a call center are usually divided into different groups serving different types of customers. Such dedicated groups can be merged into a single group through cross-training, as discussed by Tekin, Hopp, and Oyen [23].

The benefit of pooling call centers can be illustrated through the following *square-root safety staffing* rule (see, e.g., Borst, Mandelbaum, and Reiman [3]). Consider  $n \geq 2$  call centers—i.e., call center  $i$  ( $i = 1, 2, \dots, n$ ), which has an arrival rate  $\lambda_i$  and a service rate  $\mu$ . Let  $R_i = \lambda_i/\mu$  denote call center  $i$ 's *offered load*. The asymptotically optimal staffing level that balances call center  $i$ 's

cost and its customers' waiting time can be written as  $N_i = R_i + \beta\sqrt{R_i}$ , where  $\beta > 0$  is a parameter dependent on the staffing cost  $c$  and the customer's waiting cost  $a$  [3]. Assume that  $a$ ,  $c$ , and  $\mu$  are the same for all call centers, which implies that  $\beta$  is also the same across all call centers. Under the above assumption, the total staff number needed before pooling is  $\sum_{i=1}^n \lambda_i/\mu + \beta \sum_{i=1}^n \sqrt{\lambda_i/\mu}$ , and that needed after pooling is  $\sum_{i=1}^n \lambda_i/\mu + \beta\sqrt{\sum_{i=1}^n \lambda_i/\mu}$ . Since  $\sqrt{\sum_{i=1}^n \lambda_i/\mu} < \sum_{i=1}^n \sqrt{\lambda_i/\mu}$ , we find that pooling  $n$  call centers can reduce the total staff number.

Although the square-root safety staffing rule can determine the total required staff number for call centers after pooling, the rule still cannot indicate how to allocate the total staff number to each individual call center. One may note that each call center shall have an incentive to cooperate with other(s), *if and only if* the number of its staff is reduced as a result of pooling. A critical question thus arises as follows: how many staffs shall be allocated to each call center such that all centers are willing to pool?

In this note, we construct a cooperative staff allocation game for  $n$  call centers that pool their capacities. Assuming that the staffs at all call centers have similar working skills, we find that the centers have an identical service rate  $\mu$ . In addition, the staffing cost  $c$  and the customer's waiting cost  $a$  are assumed to be identical at all call centers. We show that our staffing game is essential and submodular, and thereby, convex with a non-empty core. Using the Shapley value solution concept, we derive a *unique* and *fair* staff allocation scheme, which indicates that after pooling, the number of staffs allocated to call center  $i$  is  $N_i = R_i + \beta \times \kappa_i/\sqrt{\mu}$ , where  $\kappa_i \equiv \sum_{k=1}^n \{[(k-1)!(n-k)!/(n!)] \sum_{l_i=1}^{\binom{n-1}{k-1}} (\sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i})\}$ , and  $C(l_i; k)$  is the  $l_i$ th coalition formed by  $k$  call centers inclusive of center  $i$ . We find that the Shapley value-characterized staffing rule is always in the non-empty core of our game; that is, such allocation mechanism is fair and stable, and no call center has an incentive to leave the pooling coalition.

The remainder of the note is organized as follows. In Section 2, we review the related literature. Our main results are discussed in Section 3. Moreover, a brief description of cooperative game theory—which is a major methodology used in our analysis—is presented in online Appendix A. All proofs are relegated to online Appendix B. We provide a numerical example to illustrate our cooperative game analysis in online Appendix C.

## 2 Literature Review

This note is related to the literature on the staffing and pooling problems for call centers. With diffusion approximation, Borst, Mandelbaum, and Reiman [3] considered the staffing problem of a single-class, single-pool  $M/M/N$  queue subject to cost consideration and other constraints. Mandelbaum and Zeltyn [16] extended Borst, Mandelbaum, and Reiman's analysis [3] into a model with customer abandonment. More publications along this line can be found in the surveys written by Akşin, Armony, and Mehrotra [1] and Gans, Koole, and Mandelbaum [8].

The study on call center pooling mainly focuses on workforce management. Jouini, Dallery, and Nait-Abdallah [13] examined the benefits of migrating from a call center with all agents pooled

together towards a call center with dedicated agent groups. The authors showed that, despite the drawbacks of less pooling effects, such change can bring some benefits due to better human resource management. Tekin, Hopp, and Oyen [23] investigated the benefits of pooling dedicated groups of agents into a general group after cross-training. They considered the impact of system parameters such as the number of servers, and the mean and correlation of service times, on the decision on which groups to pool. Van Dijk and van der Sluis [25] discussed the benefits of pooling call centers and showed that with multiple job types, the pooling effect could be negative. Different from the above publications, we now focus on how to fairly allocate among multiple call centers the staffing cost saving that result from pooling their capacities.

Other relevant literature includes those publications regarding the cooperative game in queueing systems. In fact, all early publications concerning the pooling of call centers are mainly related to the cooperative game analysis of fairly allocating total waiting costs among waiting customers, as discussed by, e.g., Chun ([4], [5]), Haviv [11], Haviv and Ritov [12], Katta and Sethuraman [14], Maniquet [17], etc. Applying the theory of cooperative games to address the capacity pooling problem for queueing systems starts in recent years. Anily and Haviv [2] addressed the problem of how to share the cost savings among multiple one-server service systems, showing that the core of the game is non-empty. Yu, Benjaafar, and Gerchak [29] considered a similar problem and presented a cost-sharing mechanism where each call center is better off when cooperating than acting individually. Different from Anily and Haviv [2] and Yu, Benjaafar, and Gerchak [29]—who focused on the capacity pooling among *one*-server queues, we consider the staff pooling among *multiple*-server call centers.

### 3 Cooperative Staffing Game: Model and Analysis

In this section, as discussed in online Appendix A, we first calculate the characteristic values of all possible coalitions to construct our  $n$ -center cooperative staffing game. Then we derive a unique, fair staff allocation scheme for these call centers.

#### 3.1 Characteristic Values of All Possible Coalitions

In our cooperative staffing game, the characteristic value of a coalition is defined as the number of staff that are needed for all call centers belonging to the coalition. For the empty coalition  $\emptyset$ , there is no call center and therefore, the number of staff is zero. Thus, the characteristic value of the empty coalition is  $v(\emptyset) = 0$ .

**Characteristic Values of One-Center Coalitions.** From §1, we know that before cooperating with any other call centers, the number of the staff needed by a single call center  $i$  is  $N_i = R_i + \beta \times \sqrt{R_i}$  (Borst, Mandelbaum, and Reiman [3]). Thus the characteristic value  $v(i)$ ,  $i = 1, 2, \dots, n$ , for one-center coalition  $\{i\}$  is calculated as

$$v(i) = N_i = R_i + \beta \times \sqrt{R_i}. \quad (1)$$

**Characteristic Values of  $k$ -Center Coalitions with  $2 \leq k \leq n$ .** As any subset of  $k$  ( $2 \leq k \leq n$ ) call centers may form a  $k$ -center coalition, there are  $\binom{n}{k} = n!/[k!(n-k)!]$  possible  $k$ -center coalitions. Without loss of generality, we now consider the  $l$ th  $k$ -center coalition, which we denote by  $C(l; k)$ , for  $l = 1, 2, \dots, \binom{n}{k}$  and  $2 \leq k \leq n$ . According to Borst, Mandelbaum, and Reiman [3], the characteristic value  $v(l; k)$ , which is the number of staffs needed for the coalition  $C(l; k)$ , can be computed as

$$v(l; k) = N(l; k) = R(l; k) + \beta \times \sqrt{R(l; k)}, \quad \text{for } l = 1, 2, \dots, \binom{n}{k} \text{ and } 2 \leq k \leq n. \quad (2)$$

where  $R(l; k) \equiv \sum_{i \in C(l; k)} (\lambda_i / \mu)$  denotes the offered load of the coalition  $C(l; k)$ . Note that there exists only one  $n$ -player (*grand*) coalition. We simply write the grand coalition as  $C(n) \equiv \{1, 2, \dots, n\}$ , and its characteristic value is

$$v(n) = N(n) = R(n) + \beta \times \sqrt{R(n)}, \quad \text{where } R(n) \equiv \frac{\sum_{i=1}^n \lambda_i}{\mu}.$$

## 3.2 Staffing Decision: Solution of Our Cooperative Game

To solve the staff allocation problem for the  $n$  pooled call centers, we need first examine whether there exists a staff allocation approach such that some or all call centers are willing to cooperate. More specifically, we need investigate which coalition is stable. The stability of a coalition means that no member (call center) in the coalition has an incentive to leave the coalition. Since each call center aims at reducing its staff number and saving its operating cost, a coalition in our cooperative staffing game is stable *if and only if* each call center in the coalition can hire *less* staff after joining the coalition.

A necessary condition for the stability of our  $n$ -center cooperative game  $\mathcal{G}$  is that the game is *subadditive*, that is,  $v(\mathcal{C}' \cup \mathcal{C}'') \leq v(\mathcal{C}') + v(\mathcal{C}'')$  for any two *disjoint* coalitions  $\mathcal{C}'$  and  $\mathcal{C}''$  in the game ([22]). For example, assume that there are two call centers, center 1 and center 2. Before call centers 1 and 2 cooperate, they need  $v(1)$  and  $v(2)$  staff, respectively; but, after the two centers are centralized, they jointly need  $v(12)$  staff. If  $v(12) < v(1) + v(2)$ , then we can find a staff allocation scheme that assures the stability of the coalition  $\{1, 2\}$ . Otherwise, if  $v(1) + v(2) \leq v(12)$ , then we cannot find a staff allocation scheme to assure that both centers are willing to stay together in the coalition  $\{1, 2\}$ . Hence, it is important to determine whether our game is subadditive.

### 3.2.1 Submodularity

A cost (staffing) cooperative game must be convex and subadditive if its characteristic function is submodular (Driessen [7] and Topkis [24]). We learn from Driessen [7] that our game is submodular if  $v(T \cup \{j\}) - v(T) \leq v(S \cup \{j\}) - v(S)$ , for all  $S \subseteq T \subseteq C(n) \setminus \{j\}$ .

**Theorem 1** The characteristic function of our  $n$ -player cooperative game  $\mathcal{G}$  is submodular; thus, the game  $\mathcal{G}$  is convex and subadditive. ■

The above theorem implies that when more call centers form a coalition, the characteristic value of the coalition is lower owing to the economies of scope resulting from the subadditive property of the game. That is, as a result of pooling more centers, less staff are needed and the centers should be more efficient. It thus follows that all the  $n$  centers shall have incentives to join the grand coalition  $C(n)$ .

However, for the grand coalition  $C(n)$  to be stable, we still need a sufficient condition; that is, the staff number  $v(n)$  shall be allocated to all call centers in a *fair* way. More specifically, if the number of the staff assigned to each center is no more than what this center has to hire after leaving the grand coalition, then all call centers are willing to stay in the grand coalition which makes the grand coalition stable.

### 3.2.2 Fair Staff Allocation Scheme

Denote  $m_i$  as the number of the staff allocated to center  $i$ . Then we can characterize any proper staff allocation of the characteristic value (total staff number)  $v(n)$  by using an  $n$ -tuple of numbers  $M \equiv (m_1, m_2, \dots, m_n)$  with the following two properties: (i) *individual rationality*, i.e.,  $m_i \leq v(i)$ , for  $i = 1, 2, \dots, n$ ; (ii) *collective rationality*, i.e.,  $\sum_{i=1}^n m_i = v(n)$ . A  $n$ -tuple  $(m_1, m_2, \dots, m_n)$  satisfying the above two properties is called an *imputation* for the staffing game  $\mathcal{G} = (C(n), v)$  ([22]). Below, in order to find a fair staff allocation scheme, we need first examine the non-emptiness of the *core* (Gillies [9] and Owen [19]), which, for our game, is defined as the set of all undominated imputations  $(m_1, m_2, \dots, m_n)$  such that  $\sum_{i \in T} m_i \leq v(T)$  for *all* coalitions  $T \subseteq C(n)$ .

**Theorem 2** The core for our  $n$ -player cooperative staffing game in characteristic-function form is non-empty. That is, the grand coalition  $C(n)$  is stable if all call centers implement a staff allocation scheme in the core. ■

Since any point in the non-empty core represents a fair imputation (allocation scheme), there could exist many staff assignment schemes each assuring the stability of the grand coalition. An important question thus arises: Which staff allocation scheme in the core shall be applied to allocate the total staff number  $v(n)$  among  $n$  call centers? Therefore, it would be interesting to find a *unique* staff allocation scheme for our cooperative game.

According to online Appendix A, Shapley value (Shapley [21]) and the nucleolus (Schmeidler [20]) are the two commonly-used solutions each representing a unique, fair imputation (staff allocation scheme for our staffing problem). However, to obtain the nucleolus solution, one needs to solve a *series* of linear programming (LP) problems (Wang [27]); for recent applications in the business area, see Guo, Leng, and Wang [10] and Leng and Parlar [15]. Due to the complexity of the nucleolus, we shall avoid this solution in this note. Instead, we adopt the Shapley value solution concept, under which the number of staff allocated to call center  $i$ ,  $i = 1, 2, \dots, n$ , is calculated as

$$m_i = \sum_{i \in T} \frac{(|T| - 1)!(n - |T|)! [v(T) - v(T - i)]}{n!},$$

where  $T$  denotes a possible call-center coalition that center  $i$  joins and  $|T|$  is the number of call centers in coalition  $T$ . It is proper to use Shapley value to characterize a unique, fair staff allocation scheme for our  $n$ -player cooperative game because of the following three reasons:

1. As discussed in online Appendix A, Shapley value is based on three axioms (i.e., symmetry; zero allocation to dummy player; additivity). Clearly, our staffing problem satisfies the above three axioms.
2. As discussed by Topkis [24], Shapley value for a convex cooperative game must exist in a non-empty core. As our  $n$ -player cooperative game is convex, Shapley value must be in the non-empty core of our game and the resulting staff allocation scheme can assure the stability of the grand coalition.
3. Shapley value is a *monotonic* solution (Megiddo [18] and Young [28]). For our cooperative staffing game, the monotonicity of a solution means that, if the staff number for each possible coalition decreases, then the number of the staff assigned to each center shall also decrease. Since any acceptable staff allocation scheme should be monotonic, Shapley value is a proper solution concept for our staffing problem.

**Theorem 3** When we use Shapley value to allocate the total staff number  $v(n)$  among  $n$  call centers, the number of the staff assigned to call center  $i$ ,  $i = 1, 2, \dots, n$ , is computed as

$$m_i = R_i + \beta \times \frac{\kappa_i}{\sqrt{\mu}}, \quad (3)$$

where  $R_i = \lambda_i/\mu$  is call center  $i$ 's offered load and,

$$\kappa_i \equiv \sum_{k=1}^n \left[ \frac{(k-1)!(n-k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} \left( \sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} \right) \right]. \quad \blacksquare \quad (4)$$

It is interesting to note from (3) that, when we use Shapley value to fairly allocate total staff number  $v(n)$  among  $n$  call centers, the number of staff assigned to call center  $i$  ( $i = 1, 2, \dots, n$ ) is simply equal to the center's *offered load*  $R_i$  plus an additional term (i.e.,  $\beta \times \kappa_i/\sqrt{\mu}$ ). This resembles the square-root safety staffing rule. Recall from Section 3.1 that, when call center  $i$  does not cooperate with any other centers but instead serves customers by itself, this center's optimal staff number  $v(i)$  equals the offered load  $R_i$  plus an addition term (i.e.,  $\beta \times \sqrt{R_i} = \beta \times \sqrt{\lambda_i}/\sqrt{\mu}$ ). The only difference is that the term  $\sqrt{\lambda_i}$  in the latter is replaced by  $\kappa_i$  in the former. The following theorem shows the order between them.

**Theorem 4** For call center  $i = 1, 2, \dots, n$ ,  $\kappa_i < \sqrt{\lambda_i}$ .  $\blacksquare$

Theorem 4 implies that for our  $n$ -player cooperative game, the Shapley value-characterized staff number  $m_i$  for center  $i$  is smaller than  $v(i)$ . That is, if  $n$  call centers cooperate to jointly

serve customers, then the staff number for each call center must be smaller than that when these centers operate independently. In fact, this important result confirms our Theorems 1 and 2 : If the call centers do not operate independently but decide to cooperate, then the resulting  $n$ -player cooperative game must be submodular with a non-empty core. Moreover, as discussed previously, Shapley value must exist in the non-empty core. Therefore, when we use (3) to assign staff among  $n$  call centers, all the centers are better off by cooperating with each other than by operating independently and thus the grand coalition  $C(n)$  is stable.

**Corollary 1** If  $n$  call centers cooperate to form the grand coalition  $C(n)$  instead of operating independently, call center  $i$  ( $i = 1, 2, \dots, n$ ) can reduce its staff number by  $\beta(\sqrt{\lambda_i} - \kappa_i)/\sqrt{\mu}$ , where  $\kappa_i$  is given as in (4). ■

Note that a cost cooperative game with  $n$  player is essential if  $\sum_{i=1}^n v(i) > v(n)$  ([22]). We can further conclude that our cooperative staffing game is *essential* since  $\sum_{i=1}^n v(i) > \sum_{i=1}^n m_i = v(n)$ . This property demonstrates that the pooling of  $n$  call centers can *essentially* improve the efficiency of system-wide operation.

**Remark 1** When we calculate  $v(i)$  and  $v(l; k)$  in Section 3.1 and compute the Shapley value-characterized staff number  $m_i$  by using (3), the resulting number of staff could be a decimal number rather than an integer. For such case, one may believe that we need to round that decimal number to an integer. This is unnecessary. In reality, a firm may hire both full-time and part-time staff. A decimal staff number indicates that the center hires some part-time staff. For example, if  $m_i = 2.3$ , then the call center  $i$  can hire two full-time staff and one part-time staff who works at the center for only 30% of normal working time. ◁

A numerical example is provided in online Appendix C to illustrate the above analysis.

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## References

- [1] O. Akşin, M. Armony, and V. Mehrotra. The modern call center: A multidisciplinary perspective on operations management research. *Production and Operations Management*, 16(6):655–688, November–December 2008.
- [2] S. Anily and M. Haviv. Cooperation in service systems. *Operations Research*, 58(3):660–673, May–June 2009.



- [3] S. Borst, A. Mandelbaum, and M. I. Reiman. Dimensioning large call centers. *Operations Research*, 52(1):17–34, January–February 2004.
- [4] Y. Chun. No-envy in queueing problems. *Economic Theory*, 29(1):151–162, September 2006.
- [5] Y. Chun. A pessimistic approaches to the queueing problem. *Mathematical Social Sciences*, 51(2):171–181, March 2006.
- [6] O. Donin and MOCCA team. The call center of “US bank”, 2006. Project DataMOCCA, SEESStat, Technion, Israel.
- [7] T. S. H. Driessen. *Cooperative Games, Solutions and Applications*. Kluwer Academic Publishers, Boston, 1988.
- [8] N. Gans, G. Koole, and A. Mandelbaum. Telephone call centers: Tutorial, review, and research prospects. *Manufacturing & Service Operations Management*, 5(2):79–141, Spring 2003.
- [9] D. B. Gillies. Solutions to general non-zero-sum games. In A. W. Tucker and R. D. Luce, editors, *Contributions to the Theory of Games*, pages 47–85. Princeton University Press, Princeton, New Jersey, 1959.
- [10] H. Guo, M. Leng, and Y. Wang. Interchange fee rate, merchant discount rate, and retail price in a credit card network: A game-theoretic approach. *Naval Research Logistics*, 59(7):525–551, October 2012.
- [11] M. Haviv. The Aumann-Shapley price mechanism for allocating costs in congested systems. *Operations Research Letters*, 29(5):211–215, December 2001.
- [12] M. Haviv and Y. Ritov. Externalities, tangible externalities and queueing disciplines. *Management Science*, 44(6):850–858, June 1998.
- [13] O. Jouini, Y. Dallery, and R. Nait-Abdallah. Analysis of the impact of team-based organizations in call center management. *Management Sci*, 54(2):400–414, February 2008.
- [14] A. Katta and J. Sethuraman. Cooperation in queues. CORC technical reports TR-2005-03, Columbia University, New York, 2006.
- [15] M. Leng and M. Parlar. Allocation of cost savings in a three-level supply chain with demand information sharing: A cooperative-game approach. *Operations Research*, 57(1):200–213, January–February 2009.
- [16] A. Mandelbaum and S. Zeltyn. Staffing many-server queues with impatient customers: Constraint satisfaction in call centers. *Operations Research*, 57(5):1189–1205, September–October 2009.
- [17] F. Maniquet. A characterization of the Shapley value in queueing problem. *Journal of Economic Theory*, 109(1):90–103, March 2003.

- [18] N. Megiddo. On the nonmonotonicity of the bargaining set, the kernel and the nucleolus of a game. *SIAM Journal on Applied Mathematics*, 27(2):355–358, September 1974.
- [19] G. Owen. *Game Theory*. Academic Press, New York, 2nd edition, 1982.
- [20] D. Schmeidler. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, 17:1163–1170, 1969.
- [21] L. S. Shapley. A value for  $n$ -person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games II*, pages 307–317. Princeton University Press, Princeton, 1953.
- [22] P. D. Straffin. *Game Theory and Strategy*. The Mathematical Association of America, Washington, D.C., 1993.
- [23] E. Tekin, W. Hopp, and M. Oyen. Pooling strategies for call center agent cross-training. *IIE Transactions*, 41(6):546–561, 2009.
- [24] D. M. Topkis. *Supermodularity and Complementarity*. Princeton University Press, New Jersey, 1998.
- [25] N. van Dijk and E. van der Sluis. To pool or not to pool in call centers. *Production*, 17(3):296–305, May–June 2008.
- [26] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behaviour*. Princeton University Press, Princeton, 1944.
- [27] J. Wang. *The Theory of Games*. Clarendon Press, Oxford, 1988.
- [28] H. P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2):65–72, June 1985.
- [29] Y. Yu, S. Benjaafar, and Y. Gerchak. Capacity sharing and cost allocation among independent firms in the presence of congestion, 2009. Working paper. Department of Mechanical Engineering, University of Minnesota.

## Online Supplements

### “A Fair Staff Allocation Rule for the Capacity Pooling of Multiple Call Centers”

P. Guo, M. Leng, Y. Wang

## Appendix A Brief Description of Cooperative Game Theory

Consider  $n \geq 2$  call centers pool together to reduce their operations (staffing) cost. We study how each call center decides on its staffing number after pooling. Since the theory of cooperative games is a major methodology used in our analysis of the staffing problem, we briefly discuss some important solution concepts in this theory.

Von Neumann and Morgenstern [26] develop a theory of multi-person games where various subgroups of players might join together to form coalitions. For our cooperative game where the  $n$  call centers may form different coalition structures to cooperate, we construct a game in characteristic-function form by computing the characteristic values of all possible coalitions. More specifically, for our game, if a single call center joins an one-player coalition, then the call center does not cooperate with any other call centers but only operate by itself. If two or more call centers form a multi-player coalition, then all call centers in the coalition cooperate to jointly serve their customers.

In the theory of cooperative games, the “characteristic value” of a coalition is the *minimum* amount that the coalition can attain using its own efforts only. In our note, the characteristic value of a coalition is defined as the minimum necessary staff number that is needed by all call centers in the coalition. Since any one or more call centers may form a coalition, in our  $n$ -player cooperative game, all possible coalition include the empty coalition (that does not involve any call center),  $n$  one-player coalitions and  $\binom{n}{k} = n!/[k!(n-k)!]$  possible  $k$ -player coalitions where  $2 \leq k \leq n$ . Note that, when  $k = n$ , all call centers form an  $n$ -player coalition which is also known as *grand* coalition. We have to compute the characteristic values for all the above possible coalitions; in §3.1, we provide more discussions on coalition structures and corresponding characteristic values.

After building a cooperative game, we need examine whether or not all call centers are willing to form the grand coalition; that is, we will investigate the stability of the grand coalition. This is one of the most important questions in the theory of cooperative games. In order to assure the stability of the grand coalition for our staffing game, we need find a *fair* allocation scheme that determines the staff number for each call center. For our problem, the “fairness” of an allocation scheme means that the number of staff assigned to each call center should be no more than that when the call center leaves the grand coalition. That is, under a fair allocation scheme, none of the call centers should have any incentives to deviate from the grand coalition, which implies that the fair allocation scheme is *undominated* by any other possible scheme. To find a fair allocation scheme, we apply some appropriate solution concepts from the theory of cooperative games. Leng and Parlar [15] discuss two categories of commonly-used concepts: Set-valued solution concepts and unique-valued solution concepts; and they conclude that the core (Gillies [9]), Shapley value (Shapley [21]), and

the nucleolus (Schmeidler [20]) are most common solution concepts that have been widely used in the management science/operations management field. Next, we briefly describe these three important concepts.

The *core* was first introduced by Gillies [9]. The core of an  $n$ -person *cost* cooperative game in characteristic form is defined as the set of all undominated imputations  $(x_1, x_2, \dots, x_n)$  such that for all coalitions  $T \subseteq N = \{1, 2, \dots, n\}$ , we have  $\sum_{i \in T} x_i \leq v(T)$ ; see Owen [19] for the description of the core. Although the allocation schemes suggested by the core assure stability of the grand coalition, the core could be empty for some games. Even if the core is non-empty for our game, we have to address the question of which allocation (staff assignment) scheme should be used for allocating the staff among  $n$  call centers. Therefore, after examining whether the core is empty, we also need to find a *unique* fair staff allocation scheme for our  $n$ -player staffing game.

For a  $n$ -player game ( $n \geq 2$ ), there are two commonly-used solution concepts—Shapley value and the nucleolus solution. The solution concept of Shapley value represents a unique imputation (allocation scheme)  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  where the payoffs  $x_i, i = 1, 2, \dots, n$  are distributed fairly by an outside “arbitrator”. Shapley [22] derives the unique Shapley values  $(x_1, \dots, x_n)$  as  $x_i = \sum_{i \in T} (|T| - 1)!(n - |T|)! [v(T) - v(T - i)] / n!$  where  $T$  denotes a possible coalition that player  $i$  joins, and  $|T|$  is the number of players in coalition  $T$ . The allocation scheme characterized by Shapley value is based on the following three axioms: (i) *Symmetry*: two players with symmetric roles have the same allocation. This assumes that the unique imputation *only* depends on the characteristic values of all possible coalitions. (ii) *Zero allocation to dummy player*: the allocation to a *dummy* player (who adds no value to any coalition) is zero. (iii) *Additivity*: for two cooperative games  $(N, v)$  and  $(N, w)$  which have the same player set  $N$ , an acceptable allocation scheme for players in  $N$  must be additive, i.e.,  $(v + w)(T) = v(T) + w(T)$ , for  $T \subseteq N$ .

Even though the Shapley value can be computed easily by using a formula, the Shapley value may not be in the non-empty core, thus making the grand coalition unstable. An alternative solution concept is the “nucleolus” (Schmeidler [20]), which defines an allocation scheme that minimizes the “unhappiness” of the most unhappy coalition. More specifically, for a *cost* cooperative game, denote  $e_T(\mathbf{x}) = \sum_{i \in T} x_i - v(T)$  as the excess (unhappiness) of a coalition  $T$  with an imputation  $\mathbf{x}$ . Then the nucleolus can be found as follows: (i) First consider those coalitions  $T$  whose excess  $e_T(\mathbf{x})$  is the largest for a given imputation  $\mathbf{x}$ , (ii) If possible, vary  $\mathbf{x}$  to make this largest excess smaller, (iii) When the largest excess is made as small as possible, consider the next largest excess and vary  $\mathbf{x}$  to make it as small as possible, etc. Normally, the nucleolus solution can be found by solving a sequence of linear programming problems (Wang [27]), and in general, it may be difficult to compute this solution analytically. This restricts the applications of the nucleolus.

## Appendix B Proofs

**Proof of Theorem 1.** For our cooperative staffing game, we examine the submodularity of our game by Driessen’s approach [7]. That is, we investigate if the following conditions are satisfied for

our game:  $v(S \cup \{j\}) - v(S) \geq v(T \cup \{j\}) - v(T)$ , for all  $S \subseteq T \subseteq C(n) \setminus \{j\}$ .

Assume that  $S$  and  $T$  are  $k_1$ - and  $k_2$ -centers coalitions with  $k_2 \geq k_1$ , respectively; that is,  $S = C(k_1)$  and  $T = C(k_2) = S \cup C'$  where  $C' \neq \emptyset$ . Thus, the number of staffs for coalition  $S$  and that for coalition  $T$  are computed as,

$$v(S) = \frac{\sum_{i \in S} \lambda_i}{\mu} + \beta \times \sqrt{\frac{\sum_{i \in S} \lambda_i}{\mu}} \quad \text{and} \quad v(T) = \frac{\sum_{i \in T} \lambda_i}{\mu} + \beta \times \sqrt{\frac{\sum_{i \in T} \lambda_i}{\mu}}.$$

Using the above, we have,

$$v(T) - v(S) = \frac{\sum_{i \in T-S} \lambda_i}{\mu} + \frac{\beta}{\sqrt{\mu}} \times \left( \sqrt{\sum_{i \in T} \lambda_i} - \sqrt{\sum_{i \in S} \lambda_i} \right). \quad (5)$$

When call center  $j \in C(n) - T$  joins coalition  $S$ , the number of staff needed for the resulting  $(k_1 + 1)$ -player coalition  $S \cup \{j\}$  can be calculated as,

$$v(S \cup \{j\}) = \frac{\lambda_j + \sum_{i \in S} \lambda_i}{\mu} + \frac{\beta}{\sqrt{\mu}} \times \sqrt{\lambda_j + \sum_{i \in S} \lambda_i}.$$

Similarly, when call center  $j$  joins coalition  $T$ , the number of staff for the resulting  $(k_2 + 1)$ -player coalition  $T \cup \{j\}$  is calculated as,

$$v(T \cup \{j\}) = \frac{\lambda_j + \sum_{i \in T} \lambda_i}{\mu} + \frac{\beta}{\sqrt{\mu}} \times \sqrt{\lambda_j + \sum_{i \in T} \lambda_i}.$$

It thus follows that

$$\begin{aligned} & v(T \cup \{j\}) - v(S \cup \{j\}) \\ &= \frac{\sum_{i \in T-S} \lambda_i}{\mu} + \frac{\beta}{\sqrt{\mu}} \times \left( \sqrt{\lambda_j + \sum_{i \in T} \lambda_i} - \sqrt{\lambda_j + \sum_{i \in S} \lambda_i} \right). \end{aligned} \quad (6)$$

Comparing  $v(T) - v(S)$  in (5) and  $v(T \cup \{i\}) - v(S \cup \{i\})$  in (6) yields

$$\begin{aligned} Z &\equiv [v(T \cup \{j\}) - v(S \cup \{j\})] - [v(T) - v(S)] \\ &= \frac{\beta}{\sqrt{\mu}} \times \left[ \left( \sqrt{\lambda_j + \sum_{i \in T} \lambda_i} - \sqrt{\lambda_j + \sum_{i \in S} \lambda_i} \right) - \left( \sqrt{\sum_{i \in T} \lambda_i} - \sqrt{\sum_{i \in S} \lambda_i} \right) \right]. \end{aligned}$$

Letting  $A \equiv \sum_{i \in T} \lambda_i$  and  $B \equiv \sum_{i \in S} \lambda_i$ , we re-write the above to

$$Z = \frac{\beta}{\sqrt{\mu}} \times [(\sqrt{\lambda_j + A} - \sqrt{\lambda_j + B}) - (\sqrt{A} - \sqrt{B})].$$

Noting that  $\sqrt{\lambda_j + A} \geq \sqrt{\lambda_j + B}$  and  $\sqrt{A} \geq \sqrt{B}$ , we find that the sign of the term  $[(\sqrt{\lambda_j + A} -$

$\sqrt{\lambda_j + B} - (\sqrt{A} - \sqrt{B})]$  is the same as the sign of the following expression:

$$(\sqrt{\lambda_j + A} - \sqrt{\lambda_j + B})^2 - (\sqrt{A} - \sqrt{B})^2 = 2 \left[ \lambda_j + \sqrt{AB} - \sqrt{(\lambda_j + A)(\lambda_j + B)} \right],$$

which is non-positive because  $2\sqrt{AB} \leq A + B$  and so  $\lambda_j + \sqrt{AB} \leq \sqrt{(\lambda_j + A)(\lambda_j + B)}$ . So  $Z \leq 0$  and  $v(T \cup \{j\}) - v(S \cup \{j\}) \leq v(T) - v(S)$ . Thus, our cooperative game must be submodular. ■

**Proof of Theorem 2.** This theorem follows from the submodularity of our  $n$ -player cooperative game, because, as indicated by Driessen [7] and Topkis [24], any convex game must have a non-empty core. ■

**Proof of Theorem 3.** A unique Shapley value (Shapley [21]) for center  $i$  can be computed by using  $m_i = \sum_{i \in T} (|T| - 1)!(n - |T|)! [v(T) - v(T - i)] / n!$ , where  $T$  is a possible coalition that center  $i$  may join. To obtain  $m_i$ , we first identify all possible coalitions that involve center  $i$ .

1. If center  $i$  does not cooperate with any other centers but serves customers by itself, then only coalition  $T = \{i\}$  shall be considered; thus, we have,

$$\frac{(|T| - 1)!(n - |T|)! [v(T) - v(T - \{i\})]}{n!} = \frac{v(i) - v(\emptyset)}{n} = \frac{1}{n} \left[ \frac{\lambda_i}{\mu} + \beta \times \sqrt{\frac{\lambda_i}{\mu}} \right].$$

2. We now assume that center  $i$  cooperates with other  $(k - 1)$  call centers ( $2 \leq k \leq n - 1$ ) to form a  $k$ -center coalition. Since any  $(k - 1)$  call centers among  $(n - 1)$  centers, exclusive of center  $i$ , may cooperate with center  $i$ , there are  $\binom{n-1}{k-1} = (n - 1)! / [(k - 1)!(n - k)!]$  possible  $k$ -player coalitions that center  $i$  joins. Denoting these  $\binom{n-1}{k-1}$   $k$ -player coalitions (including center  $i$ ) by  $C(l_i; k)$ , for  $l_i = 1, 2, \dots, \binom{n-1}{k-1}$  and  $2 \leq k \leq n - 1$ , we find that, if  $|T| = k \in [2, n - 1]$ , then,

$$\begin{aligned} & \frac{\sum_{i \in T} (|T| - 1)!(n - |T|)! [v(T) - v(T - \{i\})]}{n!} \\ &= \sum_{k=2}^{n-1} \left\{ \frac{(k - 1)!(n - k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} [v(l_i; k) - v(C(l_i; k) - \{i\})] \right\} \\ &= \sum_{k=2}^{n-1} \left\{ \frac{(k - 1)!(n - k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} \left[ \frac{\lambda_i}{\mu} + \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} \right) \right] \right\}. \end{aligned}$$

3. If call center  $i$  cooperates with other  $(n - 1)$  centers, then all of  $n$  centers form the grand

coalition  $C(n)$ . That is, if  $T = C(n)$ , then we have,

$$\begin{aligned} & \frac{(|T| - 1)!(n - |T|)! [v(T) - v(T - \{i\})]}{n!} \\ &= \frac{v(n) - v(C(n) - \{i\})}{n} \\ &= \frac{1}{n} \left[ \frac{\lambda_i}{\mu} + \beta \times \left( \sqrt{\frac{\sum_{j=1}^n \lambda_j}{\mu}} - \sqrt{\frac{\sum_{j=1}^n \lambda_j - \lambda_i}{\mu}} \right) \right]. \end{aligned}$$

We summarize the above, and find that the Shapley value-based staff number for call center  $i$ ,  $i = 1, 2, \dots, n$  can be calculated as,

$$\begin{aligned} m_i &= \sum_{k=1}^n \left\{ \frac{(k-1)!(n-k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} \left[ \frac{\lambda_i}{\mu} + \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} \right) \right] \right\} \\ &= \sum_{k=1}^n \left[ \frac{(k-1)!(n-k)!}{n!} \times \sum_{l_i=1}^{\binom{n-1}{k-1}} \left( \frac{\lambda_i}{\mu} \right) \right] \\ &+ \sum_{k=1}^n \left\{ \frac{(k-1)!(n-k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} \left[ \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} \right) \right] \right\}. \quad (7) \end{aligned}$$

The first term in (7) can be simplified as,

$$\sum_{k=1}^n \left[ \frac{(k-1)!(n-k)!}{n!} \times \sum_{l_i=1}^{\binom{n-1}{k-1}} \left( \frac{\lambda_i}{\mu} \right) \right] = \frac{\lambda_i}{\mu};$$

and the second term in (7) can be rewritten as

$$\sum_{k=1}^n \left\{ \frac{(k-1)!(n-k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} \left[ \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} \right) \right] \right\} = \beta \frac{\kappa_i}{\sqrt{\mu}},$$

where  $\kappa_i$  is defined in (4). This theorem is thus proved. ■

**Proof of Theorem 4.** To prove that  $\kappa_i < \sqrt{\lambda_i}$ , we need to examine the term  $\sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i}$ . Because

$$\sqrt{\lambda_i \left( \sum_{j \in C(l_i; k)} \lambda_j - \lambda_i \right)} > 0,$$

we have,

$$\sqrt{\sum_{j \in C(l_i; k)} \lambda_j} < \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} + \sqrt{\lambda_i}, \text{ or, } \sqrt{\sum_{j \in C(l_i; k)} \lambda_j} - \sqrt{\sum_{j \in C(l_i; k)} \lambda_j - \lambda_i} < \sqrt{\lambda_i}.$$

Using the above, we find that

$$\begin{aligned} \kappa_i &< \sum_{k=1}^n \left[ \frac{(k-1)!(n-k)!}{n!} \sum_{l_i=1}^{\binom{n-1}{k-1}} \sqrt{\lambda_i} \right] \\ &< \sum_{k=1}^n \left[ \frac{(k-1)!(n-k)!}{n!} \frac{(n-1)!}{(k-1)!(n-k)!} \sqrt{\lambda_i} \right] \\ &< \sum_{k=1}^n \left( \frac{1}{n} \sqrt{\lambda_i} \right) \\ &= \sqrt{\lambda_i}. \end{aligned}$$

This theorem is thus proved. ■

**Proof of Corollary 1.** If  $n$  call centers operate independently, the staff number for call center  $i$  ( $i = 1, 2, \dots, n$ ) equals  $v(i) = R_i + \beta \times \sqrt{\lambda_i}/\sqrt{\mu}$ . But, if  $n$  call centers form the grand coalition  $C(n)$ , then the Shapley value-characterized staff number for call center  $i$  is  $m_i = R_i + \beta \times \kappa_i/\sqrt{\mu}$ . Thus the reduced staff number for call center  $i$  is  $v(i) - m_i = \beta(\sqrt{\lambda_i} - \kappa_i)/\sqrt{\mu}$ . ■

## Appendix C A Numerical Example

We consider a staffing problem involving three call centers ( $i = 1, 2, 3$ ). Assume that customers' arrival rates for the three call centers are respectively,  $\lambda_1 = 100/\text{hour}$ ,  $\lambda_2 = 120/\text{hour}$ , and  $\lambda_3 = 80/\text{hour}$ . These three call centers have an identical service rate,  $\mu = 150/\text{hour}$ . Moreover, for each call center, the per hour staff cost and each customer's per hour waiting cost are  $a = \$20/\text{customer}/\text{hour}$  and  $c = \$5/\text{hour}$ , respectively. Since  $r = a/c = 4$  is smaller than 10, Borst, Mandelbaum, and Reiman [3] show that the term  $\beta$  in  $v(i)$  and  $v(l; k)$  (given in Section 3.1) can be computed as

$$\beta = \sqrt{\frac{r}{1 + r \times (\sqrt{\pi/2} - 1)}} = 1.41.$$

Next, we calculate the characteristic values of all possible coalitions for our cooperative staffing game. As discussed previously, the characteristic value of empty coalition  $\emptyset$  is zero, i.e.,  $v(\emptyset) = 0$ . According to (1), we compute the characteristic values of three one-player coalitions as

$$v(1) = \frac{\lambda_1}{\mu} + \beta \sqrt{\frac{\lambda_1}{\mu}} = 1.82, \quad v(2) = \frac{\lambda_2}{\mu} + \beta \sqrt{\frac{\lambda_2}{\mu}} = 2.06, \quad v(3) = \frac{\lambda_3}{\mu} + \beta \sqrt{\frac{\lambda_3}{\mu}} = 1.56.$$



Then, using (2), we can compute other non-empty coalitions' characteristic values as follows:

$$v(12) = \frac{\lambda_1 + \lambda_2}{\mu} + \beta \sqrt{\frac{\lambda_1 + \lambda_2}{\mu}} = 3.17, \quad v(13) = \frac{\lambda_1 + \lambda_3}{\mu} + \beta \sqrt{\frac{\lambda_1 + \lambda_3}{\mu}} = 2.74,$$

$$v(23) = \frac{\lambda_2 + \lambda_3}{\mu} + \beta \sqrt{\frac{\lambda_2 + \lambda_3}{\mu}} = 2.96; \quad v(123) = \frac{\lambda_1 + \lambda_2 + \lambda_3}{\mu} + \beta \sqrt{\frac{\lambda_1 + \lambda_2 + \lambda_3}{\mu}} = 3.99.$$

According to the above, our three-player cooperative staffing game is constructed as

$$v(\emptyset) = 0; \quad v(1) = 1.82, \quad v(2) = 2.06, \quad v(3) = 1.56;$$

$$v(12) = 3.17, \quad v(13) = 2.74, \quad v(23) = 2.96; \quad v(123) = 3.99.$$

Below we apply the formula given in Theorem 3 to allocate the grand coalition's staff number 3.99 among the three call centers. That is, the Shapley value-characterized staff number for call center 1 is computed as

$$m_1 = \frac{\lambda_1}{\mu} + \frac{1}{3} \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\lambda_1} + \sqrt{\lambda_1 + \lambda_2 + \lambda_3} - \sqrt{\lambda_2 + \lambda_3} \right)$$

$$+ \frac{1}{6} \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\lambda_1 + \lambda_2} + \sqrt{\lambda_1 + \lambda_3} - \sqrt{\lambda_2} - \sqrt{\lambda_3} \right)$$

$$= 1.33;$$

and the Shapley value-characterized staff number for call center 2 is found as

$$m_2 = \frac{\lambda_2}{\mu} + \frac{1}{3} \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\lambda_2} + \sqrt{\lambda_1 + \lambda_2 + \lambda_3} - \sqrt{\lambda_1 + \lambda_3} \right)$$

$$+ \frac{1}{6} \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\lambda_1 + \lambda_2} + \sqrt{\lambda_2 + \lambda_3} - \sqrt{\lambda_1} - \sqrt{\lambda_3} \right)$$

$$= 1.56;$$

and the Shapley value-characterized staff number for call center 3 is computed as

$$m_3 = \frac{\lambda_3}{\mu} + \frac{1}{3} \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\lambda_3} + \sqrt{\lambda_1 + \lambda_2 + \lambda_3} - \sqrt{\lambda_1 + \lambda_2} \right)$$

$$+ \frac{1}{6} \frac{\beta}{\sqrt{\mu}} \left( \sqrt{\lambda_1 + \lambda_3} + \sqrt{\lambda_2 + \lambda_3} - \sqrt{\lambda_1} - \sqrt{\lambda_2} \right)$$

$$= 1.10.$$

Comparing  $v(i)$  and  $m_i$  shows that  $m_i < v(i)$ ,  $i = 1, 2, \dots, n$ . Therefore, all call centers benefit from the centralized operation and the reduced staff numbers for each center are, respectively,

$$(\sqrt{\lambda_1} - \kappa_1)/\sqrt{\mu} = 0.49, \quad (\sqrt{\lambda_2} - \kappa_2)/\sqrt{\mu} = 0.5, \quad \text{and} \quad (\sqrt{\lambda_3} - \kappa_3)/\sqrt{\mu} = 0.46.$$

Consequently, the centers should be willing to stay in the grand coalition, which is stable.