

2-1-2013

Bertrand's paradox revisited : why Bertrand's 'solutions' are all inapplicable

Darrell Patrick ROWBOTTOM

Lingnan University, Hong Kong

Follow this and additional works at: http://commons.ln.edu.hk/sw_master



Part of the [Philosophy Commons](#)

Recommended Citation

Rowbottom, D. P. (2013). Bertrand's paradox revisited: Why Bertrand's 'solutions' are all inapplicable. *Philosophia Mathematica*, 21(1), 110-114. doi: 10.1093/phimat/nks028

This Journal article is brought to you for free and open access by the Lingnan Staff Publication at Digital Commons @ Lingnan University. It has been accepted for inclusion in Staff Publications by an authorized administrator of Digital Commons @ Lingnan University.

Bertrand's Paradox Revisited: Why Bertrand's 'Solutions' Are All Inapplicable

Darrell P. Rowbottom

Department of Philosophy

Lingnan University

darrellrowbottom@ln.edu.hk

For many years, I have agreed with the majority verdict that Bertrand showed that his chord paradox is insoluble, and thereby revealed the problematic nature of the principle of indifference. But I have now changed my mind, and will here explain why. I will make my case without using any formal mathematics. This is just as well, I think, because approaching the paradox in a formal fashion is liable to obfuscate the oversight that I identify. This oversight seems so obvious, once it is pointed out, that it is hard to imagine that it has been repeatedly missed for any other reason.

This turns out to be no victory for the defenders of the principle of indifference, however. For first, we will see why the most popular attempt to solve the paradox, due to Jaynes (1973), fails. Second, it will emerge that when it is properly understood, the paradox is considerably harder to solve than Bertrand appreciated.

Let's start with the puzzle that generates the paradox. And let's use Bertrand's own words, to make sure that we are not considering a subtly different scenario:

On trace *au hasard* une corde dans un cercle. Quelle est la probabilité pour qu'elle soit plus petite que le côté du triangle équilatéral inscrit? (Bertrand 1889: 4–5)

I translate this in the following way, which I take to be faithful (if undesirably literal):

One draws *at random* a chord in a circle. What is the probability for it to be smaller than the side of the inscribed equilateral triangle?¹

Bertrand proceeds to suggest that this question can be answered in (at least) three different ways. How so? Bertrand thinks there are (at least) three different ways to draw a chord *at random* in a circle.

Modern discussions of the paradox – as found in Gillies (2000), and Rowbottom (2011) – tend to present these answers in a formal way. But this isn't really necessary, because we can instead recognize that Bertrand just outlines three different methods for drawing chords. I'll call these 'the three ways'. To make things somewhat less abstract, we might instead think of ways to cut a cylindrical cake placed on a flat surface into two pieces, with one knife stroke, in such a way that the blade of the knife remains perpendicular to the circular surfaces. (Of course, chords have no width, whereas knives do. So I'm asking you to think about infinitely thin cuts. But you get the gist. Perfectly cylindrical cakes don't exist either, but presumably that doesn't worry you!)

¹ N.B. I take 'par hasard' to be best translated as 'by chance', and 'au hasard' to be best translated as 'at random'. A more natural translation is: 'Draw a random chord in a circle. What's the probability that it's shorter than any side of an inscribed equilateral triangle?'

The first way involves only moving the knife from left to right before making the cut. Or in other words, all possible cuts must be parallel to each other. (Note that how the cake is oriented before you start cutting is not specified. We will return to this.)

The second way involves only rotating the knife about a fixed point on the circumference of the cake. Or in other words, all possible cuts must pass through the same point on the (edge of the) circular cross-section.

The third way involves choosing some point on the circular cross-section of the cake at random – for example, by getting a child to stick a (point!) candle in it – and then making that the mid-point of the cut. But there is a special rule (implicit in the way Bertrand sets things up mathematically). Only one (pre-specified) cut can be made if the centre of the circle is the point chosen.

So the problem for the principle of indifference, allegedly, is that assigning equipossibility across possible cuts, in these three different scenarios, gives different answers to the question set. Disaster. Death to the principle of indifference, except as a measly heuristic, and any version of the logical interpretation of probability dependent on it!

But wait a minute. We began with the stipulation that ‘One draws a chord *at random* in a circle’ (or for our cake example, ‘One makes a random downward separating cut through a cylindrical cake, in one stroke, with the blade remaining perpendicular to the circular surfaces’). Is that *really* what was done in any of the three ways? I don’t think so. Rather, in each case, a chord was drawn (or a cut was made) at random *from*

a proper subset of the possible chords that might be drawn (or a proper subset of the possible cuts that might be made).

In short, the charge is that none of the methods proposed involves making a random pick from all the possible chords (or cuts). So none of proposed solutions, in effect, addresses the scenario described. To bring this into relief, imagine I posed the following question in an exam paper: “50% of rabbits are black. What is the probability that a rabbit selected at random is black?” Would I be obliged to give full marks to any answer between zero and unity (or to an answer of ‘the problem is underspecified’)? After all, a mischievous student might respond “Take the subset of black rabbits. Now pick a rabbit at random from this subset. The probability that it is black is one!” I would give the answer zero, and take the student to have willfully misinterpreted, if not misunderstood, the question. (I would admire her spirit nonetheless!) And if you think this would be fair of me, then you should also agree with my refusal to take any of the aforementioned ‘three ways’ as potentially correct answers to the question set by Bertrand.

If you still disagree with me, then it would appear, at least at first sight, that you now have infinitely many ways to draw a chord (or cut) – and thus, to apply the principle of indifference – at your disposal. But some of them look so silly, frankly, that it’s hard to imagine anyone would have taken the paradox seriously if they had been proposed in place of the ‘three ways’ above. For example, you could just specify two particular ways to cut and then have the choice between those two ways be appropriately random. Or you could opt for n ways, selected such that the probability

comes out to whatever value you want. (And now the principle of indifference is not even a heuristic.)

Of course, the ‘three ways’ all involve infinitely many possible cuts. But adding this constraint, if you think it should be added, still allows for cutting strategies that appear odd. Here’s one: cut the cake such that the ratio of the volumes of the two resulting pieces is 1:2.² And another even more striking possibility, suggested by the third way (as I’ve presented it) is simply: cut the cake through the centre. Infinitely many chords pass through the centre of a circle; and it was Bertrand’s failure to spot this that messed up his calculation for his ‘third way’.

Recognizing this, however, allows us to adjust the ‘third way’ so as to allow all possible cuts. We just remove the ‘special rule’ that only one cut may be made through the centre. And now, it seems to me, we have a pertinent way to answer the question posed. (One point, namely the central one, has an ‘angle of cut’ variable associated with it. The others don’t.) We can assign equiprobability to each possibility outlined.

This is bad news for Jaynes (1973), who argues that the first of Bertrand’s ways is the correct one (or, at least, issues in a calculation that gives the correct answer). But in order to see why, we need to consider his argument for preferring the answer arrived at by using the first way, which is more nuanced than is sometimes appreciated. In essence, Jaynes recognizes that there are infinitely many possible cuts of each length. And he reasons from there to the conclusion that considering just one set of the

² Considering the possible ratios, $n:m$, makes it apparent that there are infinitely many cutting strategies involving infinitely many possible cuts.

possible cuts, which is itself an infinite set, will give the same answer as that to the original question set. However, his point might have been made much more simply and directly than he manages with his talk of ‘rotational invariance’. Instead, consider a two-step procedure for cutting the cake. First, orient the cake at an angle of your choice, by turning it clockwise (or anti-clockwise) to any extent you desire. Then adopt the cutting procedure described in the first way, i.e. only move the knife from left to right before making the cut (such that all possible cuts, *at this point of the process*, are parallel). Now all possible chords may be selected. (You may think of this as using the principle of indifference twice; once over angles of orientation, and once over cuts perpendicular to a straight line drawn across the top of the cake before orientation occurs.)

So here’s the rub. We now clearly have two different ways of selecting all the chords; the modified version of the third way, where the angle variable is just applied to the cut through the centre, and the modified version of the first way, where (random) orientation is part of the selection process. But what makes these different from the other ways to pick a cut from the set of all possible cuts? If we restricted the problem to cuts with a rational length, for instance, one might choose two integers at random, n and m , and then choose a cut such that the ratio of the resulting surface areas of the top of the cake is $n:m$. Not restricting the problem in such a way, i.e. allowing irrational lengths, just requires a somewhat more sophisticated alternative selection process, mathematically speaking.

The conclusion is that none of Bertrand’s ‘three ways’ is a valid potential answer to the question that he sets, despite appearances to the contrary. But when the question is

properly understood, it is harder to answer, and remains possible to answer in several different ways. None of those ways, to the best of my knowledge, has been advocated as a unique solution to the paradox.³

References

Bertrand, J. 1889. *Calcul des Probabilités*. New York: Chelsea, 3rd Ed., 1960.

Gillies, D. A. 2000. *Philosophical Theories of Probability*. London: Routledge.

Jaynes, E. T. 1973. ‘The Well-Posed Problem’, *Foundations of Physics* 3, 477–93

Rowbottom, D. P. 2011. *Popper’s Critical Rationalism: A Philosophical Investigation*. London: Routledge.

Shackel, N. 2007. ‘Bertrand’s Paradox and the Principle of Indifference’, *Philosophy of Science* 74, 150–175.

³ I take this to be further support for the view that: ‘Bertrand’s paradox continues to stand in refutation of the principle of indifference’ (Shackel 2007).